Reducing subspaces of multiplication operators on function spaces

Dedicated to the Memory of Chen Kien-Kwong on the Occasion of his 120th Birthday

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Abstract. This survey presents the brief history and recent development on commutants and reducing subspaces of multiplication operators on both the Hardy space and the Bergman space, and von Neumann algebras generated by multiplication operators on the Bergman space.

§1 Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{C}^n \), \( \mathcal{H} \) be a Hilbert space consisting of some holomorphic functions over \( \Omega \). If for every \( \lambda \in \Omega \), the evaluation functional

\[ E_\lambda : f \rightarrow f(\lambda), \quad f \in \mathcal{H} \]

is bounded, then \( \mathcal{H} \) is called a reproducing kernel Hilbert space of holomorphic functions on \( \Omega \). Both the Hardy space \( H^2(\mathbb{D}) \) and the Bergman space \( L^2_a(\mathbb{D}) \) over the unit disk are classical reproducing kernel Hilbert spaces, which are the main underlying spaces we will discuss here. The Hardy space \( H^2(\mathbb{D}) \) consists of all holomorphic functions \( f \) on \( \mathbb{D} \) satisfying

\[ \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty, \]

and the Bergman space consists of all holomorphic functions over \( \mathbb{D} \) which are square integrable with respect to \( dA \), the normalized area measure on \( \mathbb{D} \). Let \( T \) be a bounded linear operator on a Hilbert space \( \mathcal{H} \). If \( M \) is a closed subspace satisfying \( TM \subseteq M \), then \( M \) is called an invariant subspace of \( T \). If in addition \( M \) also is invariant under \( T^* \), then \( M \) is called a reducing subspace of \( T \).

We consider a reproducing kernel Hilbert space \( \mathcal{H} \) of holomorphic functions on \( \Omega \). Given a bounded holomorphic function \( \phi \) on \( \Omega \), if the map \( f \mapsto \phi f, f \in \mathcal{H} \) defines a bounded operator, then denote it by \( M_\phi \), called a multiplication operator on \( \mathcal{H} \). This function \( \phi \) is called the symbol for \( M_\phi \). Recall that a von Neumann algebra is a unital \( C^* \)-algebra on a Hilbert space, which is closed in the weak operator topology. In this paper, we set \( \mathcal{W}^* (\phi) \) as the von Neumann algebra generated by \( \phi M_\phi \).
algebra generated by $M_\phi$ and write $V^*(\phi) \triangleq W^*(\phi)'$, the commutant algebra of $W^*(\phi)$. We emphasize that both $W^*(\phi)$ and $V^*(\phi)$ are von Neumann algebras.

Now we consider the reducing subspaces for $M_\phi$ from the view of von Neumann algebras. For each reducing subspace $M$, denote by $P_M$ the orthogonal projection onto $M$. Then it is easy to see that $P_M$ commutes with $M_\phi$, and hence with $M_\phi^*$. On the other hand, if $P$ is a (self-adjoint) projection in $V^*(\phi)$, then the range of $P$ is necessarily a reducing subspace for $M_\phi$. Thus there is a one-to-one and onto correspondence: $M \mapsto P_M$, which maps all reducing subspaces of $M_\phi$ to projections in $V^*(\phi)$. Since a von Neumann algebra is generated by its projections, the study of the lattice of reducing subspaces for $M_\phi$ is equivalent to that of projections in $V^*(\phi)$. Also notice that by von Neumann bicommutant theorem $V^*(\phi)' = W^*(\phi)$. This helps to understand the structures of both $V^*(\phi)$ and $W^*(\phi)$ by reducing subspaces of $M_\phi$.

There are many motivations to study the reducing subspaces of multiplication operators on reproducing kernel Hilbert spaces of holomorphic functions. First, it is to understand connections of complex analysis and von Neumann algebras; secondly, when $\phi$ is a holomorphic covering map from $D$ onto a bounded planar domain $\Omega$, we will see how both $W^*(\phi)$ and $V^*(\phi)$ are related to geometric properties of domains, and their fascinating connections to one of the long-standing problems in free group factors $L_\infty(F_n) \cong L_\infty(F_m)$ for $n \neq m$ and $n, m \geq 2$; here $F_n$ denotes the free group on $n$ generators, and $L_\infty(F_n)$ is the von Neumann algebra generated by left regular representation of $F_n$ on $l^2(F_n)$. Thirdly, it is known that the famous Invariant Subspace Problem is equivalent to the problem of the invariant subspace lattice of the Bergman space. Precisely, the Invariant Subspace Problem asks that, if $T$ is a bounded linear operator on a separable Hilbert space $H$ and $\dim H = \infty$, then must $T$ have a proper invariant subspace? It is well known that this problem is equivalent to the following: if $M$ and $N$ are $M_z$-invariant subspaces of $L^2(D)$ such that $N \subseteq M$ and $\dim M \cap N = \infty$, then is there another $M_z$-invariant subspace $L$ satisfying $N \subseteq L \subseteq M$? In general, the above problem is indeed equivalent to the problem whether there exists some single operator in the $A_\infty$-class that is saturated [7].

This survey is to recall some developments in commutants, reducing subspaces and von Neumann algebras related to multiplication operators, defined on both the Hardy space and the Bergman space.

§2 Commutants and reducing subspaces for multipliers on the Hardy space

This section mainly concerns the history and development on commutants and reducing subspaces of analytic multiplication operators on the Hardy space of the unit disk in the 1970’s. Several remarkable advances in this period were made mainly by Abrahamse and Douglas [5]; Cowen [13, 14, 15]; Baker, Deddens and Ullman [6]; Deddens and Wong [20]; Nordgren [31]; Thomson [39, 40, 41, 42], etc.

In general, it is difficult to determine the reducing subspaces of concrete operators, even on the Hardy space $H^2(D)$. On $H^2(D)$, Nordgren gave a sufficient condition for $M_\phi$ having no nontrivial reducing subspaces [31]. Also, he proved that if $\phi = h \circ \psi$ where $h \in H^\infty$ and $\psi$
is an inner function different from the Möbius transform, then $M_\phi$ admits a nontrivial reducing subspace. This essentially follows from the fact that $M_\phi$ is an isometric operator on the Hardy space whenever $\psi$ is an inner function. On the other hand, by constructing an example Abrahamse showed that Nordgren’s condition is not necessary [1]. Since each reducing subspace of $M_\phi$ is exactly the range of some projection in $\mathcal{V}^*(\phi)$, then the problem of finding reducing subspaces can be generalized to that of determining the commutant of $M_\phi$, $\{M_\phi\}'$. As mentioned in [13], when we study an operator $T$ on a Hilbert space it is interesting to consider those operators commuting with $T$, which will help in understanding the structure of the operator $T$. Unfortunately, little is known about this question in general. Shields and Wallen [34] may be the first ones who consider the commutants of subnormal operators. Later, Deddens and Wong noticed that their methods can be applied to show that if $f$ is a univalent function, then $\{M_\phi\}' = \{M_x\}' = \{M_f : f \in H^\infty(\mathbb{D})\}$ [20]. Deddens and Wong studied this problem on $H^2(\mathbb{D})$ and raised six questions as follows.

**Question 1.** Suppose that $\phi \in H^\infty(\mathbb{D})$ has inner-out factorization $\phi = \chi F$. Must it hold that $\{M_\phi\}' = \{M_x\}' \cap \{M_F\}'$?

**Question 2.** Suppose that $\phi$ is a nonconstant function in $H^\infty(\mathbb{D})$. Is the zero operator the only compact operator in the commutant of $M_\phi$?

**Question 3.** Suppose that $\phi \in H^\infty(\mathbb{D})$. Is $\{M_\phi\}' = \{M_\psi\}'$ where $\psi$ is some inner function of which $\phi$ is a function?

**Question 4.** Suppose that $\phi \in H^\infty(\mathbb{D})$. If $\{M_\phi\}' \neq \{M_x\}'$, then is $\phi = \psi \circ h$, where $\psi \in H^\infty(\mathbb{D})$ and $h$ is an inner function different from the Möbius map?

**Question 5.** Suppose that $\phi \in H^\infty(\mathbb{D})$. If $T$ commutes with $M_\phi$ on $H^2(\mathbb{D})$, then does there exist an operator $Y$ on $L^2(T)$ that commutes with $M_\phi$ defined on $L^2(T)$ and $T = Y|_{H^2(\mathbb{D})}$?

As pointed out in [20], Question 5 asks whether the commutant of $M_\phi$ on $H^2(\mathbb{D})$ can be lifted to the commutant of the minimal normal extension of $M_\phi$.

**Question 6.** Suppose that $\mathcal{F}$ is a family of inner functions. Is it true that

$$\{M_f : f \in \mathcal{F}\}' = \{M_B\}'$$

where $B$ is a Blaschke product such that each $f \in \mathcal{F}$ has the form $f = g \circ B$, with $g \in H^\infty(\mathbb{D})$.

These six questions had stimulated many further works, see [2] [31] [13] [14] [15] [39] [40] [31] [42], and [3] [4] [5] [11] [12] [10] [13] [16] [15] [30] [35] [26] [14] [43]. Question 1 is negatively answered by Abrahamse [1]. Also, as mentioned above, by constructing a counter-example Abrahamse provided a negative answer to Question 4. In [15], Cowen constructed a function $\phi$ in $H^\infty(\mathbb{D})$ such that $M_\phi$ commutes with a nonzero compact operator, and thus gave a negative answer to Question 2. Also, [13] established two theorems giving sufficient conditions on $\phi$ such that $\{M_f\}'$ contains no nonzero compact operator. Motivated by Questions 2 and 5, [14] placed much emphasis on the commutants of multiplication operators. In particular, the relation between Questions 2 and 5 was shown by [14] Theorem 1, which states that for a nonconstant $H^\infty(\mathbb{D})$-function $\phi$, if $\{M_\phi\}'$ lifts, then $M_\phi$ does not commute with any nonzero compact operators. See [14] for the definition of “lifting”, also see [8] on the problem of commutant lifting.

**Question 6** has had a satisfactory answer in [13], as follows.

**Theorem 2.1** (Cowen). Let $\mathcal{F}$ be a family of $H^\infty$-functions. If for some point $a \in \mathbb{D}$, the greatest common divisor of the inner parts of $\{h - h(a) : h \in \mathcal{F}\}$ is a finite Blaschke product,
then there is a finite Blaschke product $B$ such that on the Hardy space $H^2(D)$
\[ \bigcap_{h \in \mathcal{F}} \{M_h\}' = \{M_B\}', \]
and for each $h \in \mathcal{F}$, there is an $H^\infty$-function $\psi$ such that $h = \psi(B)$.

Special attention should be paid to the case when $\mathcal{F}$ contains exactly one member.

It is worthwhile to mention that before Cowen\cite{13}, based on Deddens and Wong’s work, Baker, Deddens and Ullman\cite{6} proved that on the Hardy space $H^2(D)$, for any entire function $f$ there is always a positive integer $k$ such that $\{M_f\}' = \{M_z^k\}'$. In some sense, this is a consequence of Theorem 2.1. Later, applying function theoretic methods, especially the techniques of local inverse and analytic continuation, Thomson gave more general conditions for when the intersection of commutants or a single commutant to be $\{M_B\}'$ for some finite Blaschke product $B$, see \cite{39, 40, 41, 42}. An easier version of Thomson’s results reads as a special case of Theorem 2.1, see \cite{39}.

**Theorem 2.2** (Thomson). If $h$ is holomorphic on a neighborhood of $\overline{D}$, then there is always a finite Blaschke product $B$ and a function $g \in H^\infty(D)$ such that $h = g \circ B$ and $\{M_h\}' = \{M_B\}'$.

**Remark 2.3.** By a careful look at the proof of the above theorem\cite{39}, Theorem 2.2 also holds on the Bergman space $L^2_a(D)$ or on the Dirichlet space $\mathcal{D}$, see \cite{26, Chapter 4} for example. However, little has been done for $\mathcal{V}(B)$ defined on $\mathcal{D}$ when $B$ is a finite Blaschke product, see \cite{12, 43}.

Cowen’s work in \cite{13} was intended to shed light on the following problem for a special class of subnormal operators: the multiplication operators defined on function spaces, especially on the Hardy space. In spirit, he developed Thomson’s techniques and extended some of Thomson’s results in more general cases. Along this line, also see \cite{14, 15}. To a certain extent, Thomson and Cowen’s work can be generalized to the Bergman space.

It is worthwhile to mention that the before-mentioned six questions can also be raised on the Bergman space. To some extent, several among them were also answered by the literatures mentioned above. However, to our best knowledge, it seems that not much was known on Question 5 in the case of the Bergman space.

For the commutants of multiplication operators, we also call the reader’s attention to \cite{39, 40, 41, 42, 13, 14, 15}, and also \cite{3, 4, 5, 9, 10, 11, 16, 18, 30, 35, 36, 44, 45}. On the Hardy space, \cite{39, 40} gave the characterization of commutant of multiplication operator with finite Blaschke product symbol. In \cite{36} Stessin and Zhu provided the generalized Riesz factorization of inner functions, applying what they obtained a new description of the commutant of multiplication operators with inner symbols. Cowen\cite{13} afforded a characterization for the commutant in the case of holomorphic covering maps. See \cite{13, 14, 15, 17} for related results on this line.

### §3 Case of the Bergman space

The Bergman-space case is much more complicated. The topic of commutants and reducing subspaces began with Zhu’s conjecture on numbers of minimal reducing subspaces of finite Blaschke product multiplication operators \cite{14} in 2000. This topic is presently experiencing a period of prosperity, as during the past ten years a lot of remarkable progress has been
made. Several notable results mushroomed in this period, exhibiting an interplay of analytical, geometrical, operator and group theoretical approaches, and relevant results are included in [21, 19, 23, 24, 25, 26, 27, 37, 38].

When we focus on the Bergman space, the case is different and difficult. In certain special cases [6, 20, 39], multiplication operators induced by inner functions play a significant role in commutant problem. On the other hand, an analogue to Thomson's theorem on commutants holds on the Bergman space. By Remark 2.3, if \( h \) is holomorphic on a neighborhood of \( \mathbb{D} \), then there is always a finite Blaschke product \( B \) and a function \( g \in H^\infty(\mathbb{D}) \) such that \( h = g \circ B \) and \( \{ M_h \} = \{ M_B \} \) [39]. In particular, \( \mathcal{V}(h) = \mathcal{V}(B) \). Thus, our attention is drawn to studying \( M_\phi \) when \( \phi \) is an inner function. By Frostman's theorem [22, p.79, Theorem 6.4], for each inner function \( \phi \), there is always a Möbius transform \( h \) such that \( h \circ \phi \) is a Blaschke product. Then such a problem is reduced to the case of \( \phi \) being a Blaschke product.

The first consideration naturally falls into those multiplication operators defined by finite Blaschke products. It is notable that the situation of the Bergman space is quite distinct from the Hardy space. By a simple analysis, one sees that the multiplication operator \( M_\eta^n (n \geq 2) \) acting on the Hardy space of the unit disk, has infinitely many minimal reducing subspaces. However, \( M_\eta^n \) acting on the Bergman space has exactly \( n \) distinct minimal reducing subspaces. More general, when \( \eta \) is an inner function, not a Möbius transform, \( M_\eta \) acting on the Hardy space \( H^2(\mathbb{D}) \) always has infinitely many minimal reducing subspaces. Then the following problem is naturally raised:

**If \( B \) is a finite Blaschke product and \( B \) is not the Möbius map, then is there a nontrivial reducing subspace for \( M_B \) defined on \( L^2_\alpha(\mathbb{D}) \)?**

A positive answer to the above problem was provided by Hu, Sun, Xu and Yu [29]; in fact, they proved further that there is a minimal reducing subspace \( M_0 \) such that \( M_B|_{M_0} \) is unitarily equivalent to the Bergman shift \( M_\eta \). In an appropriate sense, this reducing subspace turns out to be unique [27], called the **distinguished reducing subspace**. Precisely,

\[
M_0 = \text{span}\{ B^nB^n : n \geq 0 \},
\]

also see [44].

In the case of the Bergman space, and order \( B = 2 \), it was shown that \( M_B \) exactly has two distinct minimal reducing subspaces in [33] and [44] independently. Motivated by this fact, Zhu conjectured that for a finite Blaschke product \( B \) of order \( n \), there are exactly \( n \) distinct minimal reducing subspaces [11]. In fact, in the light of [38, Theorem 3.1] Zhu's conjecture holds only if \( B(z) = \phi^n \) for some Möbius transform \( \phi \). Therefore, the conjecture is modified as follows: \( M_B \) has at most \( n \) distinct minimal reducing subspaces, and the number of nontrivial minimal reducing subspaces of \( M_B \) equals the number of connected components of the Riemann surface \( S_B \) of \( B^{-1} \circ B \) on the unit disk [21]. By a Riemann surface, we mean a complex manifold of complex dimension 1, not assumed to be connected. It is known that the modified conjecture is equivalent to the assertion that \( \mathcal{V}(B) \) is abelian. As is known, \( M_B \) has a distinguished reducing subspace for a finite Blaschke product \( B \) (order \( B \geq 2 \)), and hence \( \mathcal{V}(B) \) is always nontrivial, and furthermore \( \dim \mathcal{V}(B) \leq \text{order } B \), see [20, 24, 25, 32]. In the case of order \( B = 3, 4, 5, 6, \) the modified conjecture is demonstrated in [27, 37, 23]. By using the techniques of local inverse and group-theoretic methods, it was shown in [21] that when order \( B = 7, 8 \), \( \mathcal{V}(B) \) is abelian. The latest progress is due to Douglas, Putinar and Wang [27], who provided a conclusive answer...
for the modified conjecture as follows.

**Theorem 3.1** (Douglas-Putinar-Wang). For each finite Blaschke product $B$, $\mathcal{V}^*(B)$ is abelian, and the number of nontrivial minimal reducing subspaces of $M_B$, identical with $\dim \mathcal{V}^*(B)$, equals the number of connected components of the Riemann surface $S_B$ of $B^{-1} \circ B$ on the unit disk.

It is shown in [25] that in most cases $\dim \mathcal{V}^*(B) = 2$.

However, for an infinite Blaschke product $B$, little is known about the structure of $\mathcal{V}^*(B)$. Instead of addressing the problem for general Blaschke product, we first consider two special classes of infinite Blaschke products. One kind is thin Blaschke products, and the other is covering maps. Their behaviors are so far apart. Both classes exist abundantly and share nice properties. We begin with a very basic question:

**For each infinite Blaschke product $B$, does $M_B$ always have a nontrivial reducing subspace?**

A negative answer to this question is presented in [25] by constructing a thin Blaschke product with some desired property. The construction is considerably difficult. By applying the techniques of local inverse and analytic continuation, [25] investigates the geometry of the related Riemann surface $S_B$ of $B^{-1} \circ B$. Also, it is shown in [25] that for any thin Blaschke product $B$, $\mathcal{V}^*(B)$ is abelian, and under a mild condition, $\mathcal{V}^*(B) = \mathbb{C}I$. Thus, $\mathcal{V}^*(B)$ is very simple in most cases. However, the situation becomes very complicated when $B$ is a covering map. In fact, if $\phi$ is a holomorphic covering map from $D$ onto a bounded planar domain $\Omega$, then $\phi$ will be a Blaschke product if $\Omega$ equals the unit disk minus a discrete subset $E$ of $D - \{0\}$ [11, 22, 24].

Given any bounded planar domain $\Omega$, $\pi_1(\Omega)$ is either trivial or isomorphic to the free group $\mathbb{F}_n(1 \leq n \leq \infty)$. In case $\Omega = D - E$, with $n = \#E$, the cardinality of $E$ (allowed to be infinity), $\pi_1(\Omega)$ is isomorphic to $F_n$, and thus $\mathcal{L}(\pi_1(\Omega)) = \mathcal{L}(F_n)$. By Theorem 3.2, $\mathcal{V}^*(\phi)$ has a natural and fascinating connection to one of the long-standing problems in free group factors $\mathcal{L}(F_n) \ncong \mathcal{L}(F_m)$ for $n \neq m$ and $n, m \geq 2$. For the details, see [24] [26].

Also, we mention that Theorem 3.2 can also be generalized to regular holomorphic branched covering maps, which has a close connection with orbifold manifold, for details see [24]; For more consideration on this line, see [26] and [26] Chapter 6. It is worthwhile to mention that Abrahamse and Douglas [5] constructed a class of subnormal operators related to multiply-connected domains, and considered the von Neumann algebras generated by such operators which act on vector-valued Hardy space. However, the techniques and ideas developed in [5] are quite different from that in [24].
§4 Some further questions

This section contains some further conjectures and questions, which mainly come from [26, Chapter 7].

The last section mainly deals with the structure of $V^*(\phi)$ where $\phi$ runs throughout three classes: finite Blaschke products, thin Blaschke products and bounded holomorphic covering maps. It turns out that the representations for those operators in $V^*(\phi)$ have similar forms in all three cases, while the structures of $V^*(\phi)$ are distinct from one another. On the other hand, even if we focus only on Blaschke products, only three restricted forms are concerned: finite or thin Blaschke products, covering maps (which turn out to be interpolating Blaschke products in this case). Thus, it seems that only a very “tiny” part of all Blaschke products has been discussed. Inspired by the results in Section 3, the following question is natural [26].

**Question 4.1.** For an infinite Blaschke product $B$, is there a connection between the commutativity of $V^*(B)$ on $L^2_D$ and the density of $Z(B)$? If so, how to describe it?

Notice that $H^\infty(D)$ is a Banach space with the supremum norm, and for any metric space the notion of first category makes sense. Recall that any countable union of nowhere dense sets is called a set in the first category. The following was first raised in [26].

**Conjecture 4.2.** Except for a subset of $H^\infty(D)$ in the first category, $M_{\phi}$ acting on the Bergman space, has no nontrivial reducing subspace. Also, in an appropriate sense, for “most” infinite Blaschke products $B$, $M_B$ has not nontrivial reducing subspace.

Inspired by Theorems 3.1 and 3.2, we make the following conjecture [26].

**Conjecture 4.3.** For each Blaschke product $B$, $V^*(B)$ defined on $L^2_D$ is a finite von Neumann algebra.

A bolder conjecture is the following:

**Conjecture 4.4.** Suppose that $\phi \in H^\infty(D)$ satisfying $Z(\phi) \cap D \neq \emptyset$. If $M$ is a reducing subspace of $M_{\phi}$ such that $\dim M \cup \phi M < \infty$, then $P_M$ lies in the center of $V^*(\phi)$.
Notice that if Conjecture 4.4 holds, then under the same assumption the von Neumann algebra $P_M V^*(\phi)|_M$ is abelian. This follows from an easy observation: any orthogonal projection in $P_M V^*(\phi)|_M$ has the form $P_N$, where $N$ is a reducing subspace satisfying $N \subseteq M$, and thus $\dim N \cap \phi N \leq \dim M \cap \phi M < \infty$.

By Conjecture 4.4 all $P_N \in Z(V^*(\phi))$. Then $P_M V^*(\phi)|_M$ is a $*$-subalgebra of $Z(V^*(\phi))$, and hence it is abelian. Special interest is focused on the case when $M$ equals the whole space $L^2_a(\Omega)$. If $\phi$ is a finite Blaschke product, then Conjecture 4.4 holds, see Theorem 3.1.

Write $\Phi = (\phi_1, \cdots, \phi_n)$, where $\phi_j (1 \leq j \leq n)$ are bounded holomorphic functions on a domain $\Omega$ in $\mathbb{C}^d$. One may also consider a tuple $(M_{\phi_1}, \cdots, M_{\phi_n})$ (in short, $M_\Phi$) of multiplication operators. We set $W^*(\Phi)$ as the von Neumann algebra generated by $M_\Phi$ and write $V^*(\Phi) \doteq W^*(\Phi)'$, the commutant algebra of $W^*(\Phi)$. The following question is natural and basic.

**Question 4.5.** Suppose that $\Phi = (\phi_1, \cdots, \phi_n)$ acts on a Hardy space $H^2(\Omega)$ or Bergman space $L^2_a(\Omega)$, where $\Omega$ is a reasonable domain in $\mathbb{C}^d$. If $d > n$, is $\dim V^*(\Phi) = \infty$? If $d < n$, under what conditions do we have $V^*(\Phi) = \mathbb{C}I$? Also consider the structure of $V^*(\Phi)$ in the case of $d = n$.

Little is known about such questions for $d \geq 2$. The case $d = n$ may be of most interest. To close this section, we present an example from [26].

**Example 4.6.** Consider the permutation groups $S_n (n \geq 3)$. Write

$$\phi_1 = \sum_{1 \leq j \leq n} z_j,$$
$$\phi_2 = \sum_{1 \leq i < j \leq n} z_i z_j,$$
$$\phi_n = z_1 z_2 \cdots z_n.$$ Set $\Omega = \mathbb{B}_n$ or $\mathbb{D}^n$, and write $\Phi = (\phi_1, \cdots, \phi_n)$. Then the deck transformation group $G(\Phi)$ of $\Phi$ is isomorphic to $S_n$, and the von Neumann algebra $V^*(\Phi)$ on $L^2_a(\Omega)$ is $*$-isomorphic to $L(S_n)$. For the details, see [26].

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**References**


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