On the $p$-essential normality of principal submodules of the Bergman module on strongly pseudoconvex domains

Ronald G. Douglas, Kunyu Guo, Yi Wang

July 28, 2018

Abstract

In this paper, we show that under a mild condition, a principal submodule of the Bergman module on a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ is $p$-essentially normal for all $p > n$. This is a significant improvement of the results of the first author and K. Wang in [7], where the same result is shown to hold for polynomial-generated principal submodules of the Bergman module on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. As a consequence of our main result, we prove the submodule of $L^2_\alpha(\mathbb{B}_n)$ consisting of functions vanishing on a pure analytic subsets of codimension 1 is $p$-essentially normal for all $p > n$.

1 Introduction

Let $\mathbb{C}[z_1, \ldots, z_n]$ be the ring of analytic polynomials of $n$ variables. For a Hilbert space $\mathcal{H}$, a homomorphism

$$\Phi : \mathbb{C}[z_1, \ldots, z_n] \to \mathcal{B}(\mathcal{H})$$

defines a $\mathbb{C}[z_1, \ldots, z_n]$-module structure on $\mathcal{H}$. In this case we say $\mathcal{H}$ is a Hilbert module (over $\mathbb{C}[z_1, \ldots, z_n]$). A closed subspace $P \subset \mathcal{H}$ invariant under the module action is called a submodule. It naturally inherits a Hilbert module structure by restriction. Algebraically it is easy to see that the quotient space $\mathcal{H}/P$ also has a

2010 Mathematics Subject Classification. 47A13, 32A50, 30H20, 32D15, 47B35

Key words and phrases. Arveson-Douglas Conjecture, Complex harmonic analysis, Bergman spaces, strongly pseudoconvex domains
module structure. Since our objects are Hilbert spaces we prefer to consider it as \( Q = P^\perp \subseteq \mathcal{H} \). And then the module action is given by compression.

If the commutators \( [\Phi(z_i), \Phi(z_j)]^* \) are compact for all \( i, j \) then we say \( \mathcal{H} \) is essentially normal. If moreover the commutators are in the Schatten \( p \) class \( S^p \) for some \( p \geq 1 \), then we say \( \mathcal{H} \) is \( p \)-essentially normal.

A well known example of essentially normal Hilbert module is the Bergman module on bounded strongly pseudoconvex domains with smooth boundary in \( \mathbb{C}^n \). In fact it is \( p \)-essentially normal for all \( p > n \) [4]. We denote the module actions by \( M_p \) since they are just multiplications by the polynomials \( p \). As a convention we also use \( R_p \) to denote module actions on submodules and \( S_p \) for quotient modules. For the index functions \( z_i \) we simply write \( M_i, R_i \) and \( S_i \) for convenience.

In [1] Arveson conjectured that all submodules obtained by taking closure of a homogenous polynomial ideal in the Drury-Arveson module on the unit ball \( \mathbb{B}_n \) is \( p \)-essentially normal for all \( p > n \). Later the first author extended the conjecture to Bergman space on the unit ball and to the case of quotient modules [4]. This is usually called the Arveson-Douglas Conjecture. A lot of work has been done on this conjecture, for example, [7][10][11][13][14][15][16][22] and many others.

In particular, in [7], the first author and K. Wang proved the surprising result that a principal submodule generated by any polynomial in the Bergman space on the unit ball \( \mathbb{B}_n \) is \( p \)-essentially normal for all \( p > n \). Later Fang and Xia [11] extended this result to more general spaces, including the Hardy space on \( \mathbb{B}_n \). These results suggest that the conjecture might be true under a more general setting.

After [7], it is natural to consider non-polynomial generated principal submodules. However, the extension is not easy because the estimations in [7] are very technical and depend on the degree of the generating polynomial \( p \). We observed that the essential normality of the principal submodule can be obtained using only a key estimation ([7] Lemma 3.2) and a rather standard trick in dealing with commutators (see the proof of Theorem 4.1). To generalize the key estimation, we reformulate it into the following form and prove it in more generality, using new techniques.

**Theorem 1.1** (Corollary 3.3). Suppose \( \Omega \subseteq \mathbb{C}^n \) is a bounded strongly pseudoconvex domain with smooth boundary, \( h \) is a holomorphic function defined on a neighborhood of \( \Omega \). Then there exists a constant \( N > 0 \) such that \( \forall w, z \in \Omega \) and \( \forall f \in \text{Hol}(\Omega) \),

\[
|h(z)f(w)| \leq \frac{F(z, w)^N}{r(w)^{N+n+1}} \int_{E(w, 1)} |h(\lambda)||f(\lambda)|dv_{\lambda}(\lambda). \tag{1.1}
\]

Here the set \( E(w, 1) \) is the ball at \( w \), under the Kobayashi metric, of radius 1. The functions \( F(z, w) \) and \( r(w) \) will be introduced in the next section. In the case
of the unit ball \(B_n\), the Kobayashi metric coincides with the Bergman metric and one can take \(F(z, w) = |1 - \langle z, w \rangle|, \ r(w) = 1 - |w|^2\).

Then we are able to prove the following main theorem.

**Theorem 1.2** (Theorem 4.1). Suppose \(\Omega \subseteq C^n\) is a bounded strongly pseudoconvex domain with smooth boundary, \(h \in Hol(\Omega)\), then the principal submodule of the Bergman module \(L^2_\Omega(\Omega)\) generated by \(h\) is \(p\)-essentially normal for all \(p > n\).

The proof of Theorem 1.1 requires some effort. From our refined form (1.1) one observes that this estimation depends essentially on the behavior of \(h\) at points close to the boundary of \(\Omega\). This inspired us to seek first for a proof of inequality (1.1) for \(w\) in a neighborhood of some boundary point \(\zeta \in \partial \Omega\), and then get a global estimation by compactness. We will spend most of section 3 to prove the following local version of Theorem 1.1.

**Theorem 1.3** (Theorem 3.1). Suppose \(\Omega \subseteq C^n\) is a bounded strongly pseudoconvex domain with smooth boundary, \(\zeta \in \partial D\), \(h\) is a holomorphic function defined in a neighborhood \(U\) of \(\zeta\). Then there exist a neighborhood \(V\) of \(\zeta\) and constants \(\delta > 0, \ N > 0\) such that \(\forall w \in V \cap \Omega, \forall z \in B(w, \delta) \cap U\) and \(\forall f \in Hol(E(w, 1))\),

\[ |h(z)f(w)| \leq \frac{F(z, w)^N}{r(w)^{N+n+1}} \int_{E(w, 1)} |h(\lambda)||f(\lambda)|dv_n(\lambda). \]

Here \(B(w, \delta)\) is the Euclidean ball centered at \(w\) with radius \(\delta\).

The presence of arbitrary holomorphic function \(f\) in this inequality would dramatically increase the difficulty in proving it. To tackle this, we modify our inequality into a slightly stronger form where we first put a logarithm inside the integral sign and then obtain the original inequality by applying the Jensen’s inequality. This would single out the terms involving \(f\). In fact, using induction on the dimension, we will show that

\[
\log |h(z)| \leq \frac{1}{v(P_w(\frac{4}{n}\delta(w)^{1/2}, \frac{4}{n}\delta(w)))} \int_{P_w(\frac{4}{n}\delta(w)^{1/2}, \frac{4}{n}\delta(w))} \log |h(\lambda)|dv_n(\lambda)
\]

\[+N \log \frac{F(z, w)}{|r(w)|} + C, \]

here \(P_w(\frac{4}{n}\delta(w)^{1/2}, \frac{4}{n}\delta(w))\) is a ellipsoid comparable with the Kobayashi ball \(E(w, 1)\). The induction steps involve a parameterized version of the Weierstrass Preparation Theorem (Lemma 3.9).

In the case when \(\Omega = B_n\), more results can be obtained. First, we prove a similar characterisation as in [7] for functions in the principal submodule \([h]\). Namely,

\([h] = \{hf \in L^2_\Omega(B_n) : f \in Hol(B_n)\}\).
Moreover, for a pure \((n-1)\)-dimensional analytic subset \(V\) of an open neighborhood of \(\overline{B_n}\), we show that \(V\) has a global minimal defining function \(h\). Therefore

\[
P_V := \{ f \in L^2_a(B_n) : f|_V = 0 \} = [h].
\]

This gives us the following result on a geometric version of the Arveson-Douglas Conjecture [6][9][8].

**Theorem 1.4** (Theorem 5.10). Suppose \(V\) is a pure \((n-1)\)-dimensional analytic subset of an open neighborhood of \(\overline{B_n}\), then \(V\) has a minimal defining function \(h\) on an open neighborhood of \(\overline{B_n}\). Moreover,

\[
P_V := \{ f \in L^2_a(B_n) : f|_{V \cap B_n} = 0 \} = [h].
\]

Therefore the submodule \(P_V\) is \(p\)-essentially normal for all \(p > n\).

Results on the essential spectrum of the Toeplitz algebra on the quotient module is also obtained.

This paper is organized as follows. In section 2 we introduce notions and tools involving strongly pseudoconvex domains. In section 3 we prove the key estimation using the techniques mentioned above. In section 4 we prove our main result using the key estimation. In section 5 we obtain some further results on the unit ball \(B_n\).

The third author would like to thank professor Kai Wang in Fudan University for discussing with her.

## 2 Preliminaries

In this section we introduce some notions and tools involving strongly pseudoconvex domains. Our definitions and lemmas come from [12][19][20].

**Definition 2.1.** For \(\Omega\) a bounded domain in \(\mathbb{C}^n\) with smooth boundary, we call \(r(z)\) a defining function for \(\Omega\) provided

(1) \(\Omega = \{ z \in \mathbb{C}^n : r(z) < 0 \}\) and \(r(z) \in C^\infty(\mathbb{C}^n)\).

(2) \(|\text{grad } r(z)| \neq 0\) for all \(z \in \partial \Omega\).

For \(\Omega\) a bounded strongly pseudoconvex domain with smooth boundary we mean that there is a defining function \(r \in C^\infty(\mathbb{C}^n)\) and a constant \(k\) such that

\[
\sum_{i,j=1}^{n} \frac{\partial^2 r(p)}{\partial z_i \partial \overline{z}_j} \xi_i \overline{\xi}_j \geq k|\xi|^2
\]

for all \(p \in \partial \Omega\) and \(\xi \in \mathbb{C}^n\).
For a point \( p \in \partial \Omega \), the **complex tangent space** (cf. [19]) at \( p \) is defined by

\[
T^c_p(\partial \Omega) = \{ \xi \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial r(p)}{\partial z_j} \xi_j = 0 \}. 
\]

For \( \Omega \) a bounded strongly pseudoconvex domain with smooth boundary in \( \mathbb{C}^n \), there is a \( \delta > 0 \) such that if \( z \in \Omega_\delta := \{ z \in \Omega : d(z, \partial \Omega) < \delta \} \), then there exists an unique point \( \pi(z) \in \partial \Omega \) with \( d(z, \pi(z)) = d(z, \partial \Omega) \). The complex normal (tangent) direction at \( z \) means the corresponding directions at \( \pi(z) \). For \( z \in \Omega_\delta \), we let \( P(z, r_1, r_2) \) denote the polydisc centered at \( z \) with radius \( r_1 \) in the complex normal direction and radius \( r_2 \) in each complex tangential direction.

**Notations:** We use the notations \( \approx \), \( \lesssim \) and \( \gtrsim \) to denote relations “up to a constant (constants)” between positive scalars. For example, \( A \approx B \) means there exists \( 0 < c < C \) such that \( cB \leq A \leq CB \). \( A \lesssim B \) means there exists a constant \( C > 0 \) so that \( A \leq CB \). For a point \( z \in \Omega \), denote \( \delta(z) = d(z, \partial \Omega) \), where \( d \) is the Euclidean distance. In the case when \( \Omega \) is the unit ball \( B_n \), \( \delta(z) \) is just \( 1 - |z| \). We use the notation \( \mathbb{D} \) for the open unit disc in \( \mathbb{C} \) and \( \Delta(\lambda, r) \) for the 1-dimensional disc centered at \( \lambda \) with radius \( r \). We use \( B(z, r) \) for higher dimensional Euclidean balls centered at \( z \) with radius \( r \).

For positive integer \( k \), we use \( v_k \) to denote the Lebesgue measure on \( \mathbb{C}^k \).

**Lemma 2.2.** [19, Lemma 8] Let \( \Omega \) be a bounded strongly pseudoconvex domain with smooth boundary. Fix any defining function \( r \), then for \( z \) in a neighborhood of \( \Omega \) we have

\[
|r(z)| \approx \delta(z). 
\]

For this reason, in most of our discussions, using either \( |r(z)| \) or \( \delta(z) \) does not make a difference. We will choose whichever is more convenient.

**Definition 2.3.** Let \( \Omega \subseteq \mathbb{C}^n \) be a bounded strongly pseudoconvex domain with smooth boundary. The **Bergman space** \( L^2_{\Omega}(\Omega) \) consists of all holomorphic functions on \( \Omega \) which are square integrable with respect to the Lebesgue measure \( v_n \).

\[
L^2_{\Omega}(\Omega) = \{ f \in Hol(\Omega) : \int_{\Omega} |f(z)|^2 \, dv_n(z) < \infty \}. 
\]

For a positive integer \( l \), one defines the **weighted Bergman space** \( L^2_{\omega,l}(\Omega) \) in a similar way.

\[
L^2_{\omega,l}(\Omega) = \{ f \in Hol(\Omega) : \int_{\Omega} |f(z)|^2 |r(z)|^l \, dv_n(z) < \infty \}. 
\]
Standard argument shows that the Bergman and weighted Bergman spaces are reproducing kernel Hilbert spaces. We use $K(z, w)$ and $K_l(z, w)$ to denote their reproducing kernels, i.e.,

$$f(z) = \int_\Omega f(w)K(z, w) dv_n(w), \quad \forall f \in L^2_0(\Omega)$$

and

$$f(z) = \int_\Omega f(w)K_l(z, w)|r(w)|^l dv_n(w), \quad \forall f \in L^2_0(\Omega).$$

Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, $p \in \Omega$ and $\xi \in \mathbb{C}^n$, the infinitesimal Kobayashi metric (cf. [19][17][18]) of $\Omega$ is defined by

$$F_K(p, \xi) = \inf\{\alpha > 0 : \exists f \in \mathcal{U}(\Omega) \text{ with } f(0) = p \text{ and } f'(0) = \xi/\alpha\},$$

here $\mathcal{U}(\Omega)$ denotes the set of all holomorphic mappings from the open unit disc $\mathbb{D}$ to $\Omega$. For any $C^1$ curve $\gamma(t) : [0, 1] \rightarrow \Omega$, we define the Kobayashi length of $\gamma$ as

$$L_K(\gamma) = \int_0^1 F_K(\gamma(t), \gamma'(t)) dt.$$

If $p, q \in \Omega$, we write $\beta(p, q) = \inf\{L_K(\gamma)\}$ where the infimum is taken over all $C^1$ curves with $\gamma(0) = p$ and $\gamma(1) = q$. Then $\beta(p, q)$ is a complete metric and gives the usual topology on $\Omega$.

For $w \in \Omega$ and $r > 0$, denote $E(w, r)$ to be the Kobayashi ball

$$E(w, r) = \{z \in \Omega : \beta(z, w) < r\}.$$

Then we have the following lemma.

**Lemma 2.4.** [19 Lemma 6] Let $\Omega$ be a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$. If $z \in \Omega_\delta := \{z \in \Omega : d(z, \partial \Omega) < \delta\}$ and $\delta$ is small enough, there are constants $a_i$ and $b_i$, $i = 1, 2$ only depending on $r$ and $\Omega$ such that

$$P_w(a_1\delta(w), b_1\delta(w)^{1/2}) \subseteq E(w, r) \subseteq P_w(a_2\delta(w), b_2\delta(w)^{1/2}).$$

Here $d$ is the Euclidean distance. In particular, $v_\alpha(E(w, r)) \approx \delta(w)^{n+1}$.

Fix some defining function $r(z)$ of $\Omega$. Let

$$X(z, w) = -r(z) - \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j}(w_j - z_j)$$

(2.1)

$$-1/2 \sum \frac{\partial^2 r(z)}{\partial z_j \partial z_k}(w_j - z_j)(w_k - z_k).$$

(2.2)
And
\[ F(z, w) = |r(z)| + |r(w)| + |\text{Im}X(z, w)| + |z - w|^2. \] (2.3)

Let
\[ \rho(z, w) = |z - w|^2 + \sum_{j=1}^{n} \left| \frac{\partial r(z)}{\partial z} (w_j - z_j) \right|. \]

**Lemma 2.5 ([12][19])**. Let \( \Omega \) be a bounded strongly pseudoconvex domain with smooth boundary, then
\[ |X(z, w)| \approx |r(z)| + |r(w)| + \rho(z, w) \approx F(z, w) \]
in a region
\[ R_\delta := \{(z, w) \in \overline{\Omega} \times \overline{\Omega} : |r(z)| + |r(w)| + |z - w| < \delta \}, \]
for some \( \delta > 0 \).

**Lemma 2.6 ([20] Theorem 2.3)**. Let \( \Omega \) be a bounded strongly pseudoconvex domain with smooth boundary in \( \mathbb{C}^n \). Write \( \Gamma = \{(z, z) : z \in \partial \Omega \} \). Let \( l \) be any positive integer. Then there exists a kernel \( G_l(z, w) \) such that:

(i) \( G_l \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Gamma) \), \( G_l \) is holomorphic in \( z \).

(ii) \( G_l \) reproduces the holomorphic functions in \( L^2_{a, l}(\Omega) \); i.e., for \( f \in L^2_{a, l}(\Omega) \),
\[ f(z) = \int_{\Omega} G_l(z, w) f(w) |r(w)|^l dv_a(w). \]

(iii) \( |G_l(z, w)| \approx |X(z, w)|^{-(n+1+l)} \) for \( (z, w) \in R_\delta \) for some \( \delta > 0 \).

**Lemma 2.7 ([19])**. There exists a \( \delta > 0 \) such that when \( (z, w) \in R_\delta \),
\[ |K(z, w)| \approx |X(z, w)|^{-(n+1)}. \]
Moreover, \( K(z, w) \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Gamma) \)

In particular, \( |K(z, w)| \) and \( |G_l(z, w)| \) are uniformly bounded for \( (z, w) \notin K_\delta \), for any \( \delta > 0 \), since \( K_\delta \) is a neighborhood of \( \Gamma \). Let \( \delta > 0 \) be so small that Lemma 2.5, 2.6 and 2.7 hold on \( K_\delta \). Notice that the function \( F(z, w) \) is continuous and non-zero off the set \( \Gamma \), we have
\[ |K(z, w)| \lesssim F(z, w)^{-(n+1)} \]
and
\[ |G_l(z, w)| \lesssim F(z, w)^{-(n+1+l)} \]
for all pairs \( (z, w) \in \Omega \times \Omega \).
Lemma 2.8. There exists $\delta > 0$ such that for $(z, w) \in R_\delta$

$$\rho(z, w) \approx \rho(w, z)$$

and

$$|X(z, w)| \approx |X(w, z)|.$$ 

Proof. By definition,

$$|\rho(z, w) - \rho(w, z)| \leq \left| \sum_j \left( \frac{\partial r(z)}{\partial z_j} - \frac{\partial r(w)}{\partial z_j} \right) (w_j - z_j) \right| \leq |w - z|^2 \leq \min\{\rho(z, w), \rho(w, z)\}. $$

From this it is easy to see that

$$\rho(z, w) \approx \rho(w, z).$$

The estimation for $X$ follows immediately from this and Lemma 2.5.

Lemma 2.9. Suppose $r(z)$ and $r'(z)$ are two defining functions for $\Omega$, let $X$ and $X'$ be defined as in (2.1) for $r(z)$ and $r'(z)$. Then there exists $\delta > 0$ such that for $(z, w) \in K_\delta$

$$|X(z, w)| \approx |X'(z, w)|.$$ 

Proof. By Lemma 2.5 there exists $\delta > 0$ so that when $(z, w) \in K_\delta$

$$|X(z, w)| \approx |r(z)| + |r(w)| + \left| \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} (w_j - z_j) \right| + |z - w|^2.$$ 

Since $|r(z)| \approx \delta(z) \approx |r'(z)|$, the only part we need to take care of is

$$\left| \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} (w_j - z_j) \right|$$

and the corresponding one for $r'$. Notice that

$$\left| \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} (w_j - z_j) - \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} (w_j - z_j) \right| \leq \delta(z).$$

We can replace the derivatives at $z$ by those at $\pi(z)$. But since $r$ and $r'$ are both defining functions for $\Omega$, their gradients on the boundary points vary by a constant multiple with absolute value uniformly bounded above and away from 0 (this follows from the compactness of $\partial \Omega$). From this it is easy to see that the above quantities are equivalent. 

□
The following lemma comes from the proof of [19 Theorem 12].

**Lemma 2.10.** Fix some $r > 0$, then for $z, w \in \Omega$ and $\beta(z, w) < r$,

$$|r(z)| \approx |r(w)|.$$  

**Lemma 2.11.** Fix some $r > 0$, then there exists $\delta > 0$, for $z, w, \lambda \in \Omega$ such that $(z, \lambda), (w, \lambda) \in K_\delta$ and $\beta(z, w) < r$,

$$|X(z, \lambda)| \approx |X(w, \lambda)|.$$  

As a consequence, $F(z, \lambda) \approx F(w, \lambda)$ for all $w, z, \lambda \in \Omega$ and $\beta(z, w) < r$.

**Proof.** First, by Lemma 2.4 and Lemma 2.10, there exists $\delta > 0$ such that

$$|X(z, \lambda)| \approx |r(z)| + |r(\lambda)| + \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j} (\lambda_j - z_j) \right| + |z - \lambda|^2$$

and

$$|X(w, \lambda)| \approx |r(w)| + |r(\lambda)| + \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j} (\lambda_j - w_j) \right| + |w - \lambda|^2.$$  

for pairs $(z, \lambda), (w, \lambda) \in K_\delta$. By Lemma 2.4 and Lemma 2.10

$$|z - w|^2 \leq |r(w)| \approx |r(z)|.$$  

So

$$|z - \lambda|^2 \leq \left( |z - w| + |w - \lambda| \right)^2 \leq |z - w|^2 + |w - \lambda|^2 \leq |r(w)| + |w - \lambda|^2.$$  

Therefore, we have

$$\left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j} (z_j - \lambda_j) \right|$$

$$\leq \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial z_j} (w_j - \lambda_j) \right| + \left| \left( \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j} - \frac{\partial r(w)}{\partial z_j} \right) (z_j - \lambda_j) \right| + \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial z_j} (z_j - w_j) \right|$$

$$\leq \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial z_j} (w_j - \lambda_j) \right| + |z - w||z - \lambda| + |r(w)|$$

$$\leq \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial z_j} (w_j - \lambda_j) \right| + |z - w|^2 + |z - \lambda|^2 + |r(w)|$$

$$\leq \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial z_j} (w_j - \lambda_j) \right| + |r(w)| + |w - \lambda|^2$$

$$\leq |X(w, \lambda)|.$$  

9
Altogether we have

$$|X(z, \lambda)| \leq |X(w, \lambda)|.$$ 

Since the role of $z$ and $w$ are symmetric, we get

$$|X(z, \lambda)| \approx |X(w, \lambda)|.$$ 

This completes the proof. 

Lemma 2.12. [20] Lemma 2.7] Let $\Omega$ be a bounded strongly pseudoconvex domain with smooth boundary. Let $a \in \mathbb{R}$, $\nu > -1$, then

$$\int_{\Omega} \frac{|r(w)|^\nu}{F(z, w)^{n+1+\nu+a}} dv_n(w) \approx \begin{cases} 
1 & \text{if } a < 0 \\
\log |r(z)|^{-1} & \text{if } a = 0 \\
|r(z)|^{-a} & \text{if } a > 0
\end{cases}$$

3 An Inequality

The following theorem plays a key role in the proof of the main result.

Theorem 3.1. Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, $\zeta \in \partial D$, $h$ is a holomorphic function defined in a neighborhood $U$ of $\zeta$. Then there exist a neighborhood $V$ of $\zeta$ and constants $\delta > 0$, $N > 0$ such that $\forall w \in V \cap \Omega$, $\forall z \in B(w, \delta) \cap U$ and $\forall f \in \text{Hol}(E(w, 1))$,

$$|h(z)f(w)| \leq \frac{|X(z, w)|^N}{|r(w)|^{N+n+1}} \int_{E(w, 1)} |h(\lambda)||f(\lambda)| dv_n(\lambda).$$

Remark 3.2. It turns out that from the proof of Theorem 3.1, the requirements that $z$ being close to $w$ and that $w$ being close to the boundary is not essential. In fact, from the proof of Theorem 3.1 one obtains the following.

Corollary 3.3. Suppose $\Omega \subseteq \mathbb{C}^n$ is a bounded strongly pseudoconvex domain with smooth boundary, $h$ is a holomorphic function defined on a neighborhood of $\overline{\Omega}$. Then there exists a constant $N > 0$ such that $\forall w, z \in \Omega$ and $\forall f \in \text{Hol}(\Omega)$,

$$|h(z)f(w)| \leq \frac{F(z, w)^N}{|r(w)|^{N+n+1}} \int_{E(w, 1)} |h(\lambda)||f(\lambda)| dv_n(\lambda).$$

Before proving Theorem 3.1 and Corollary 3.3 we establish a few lemmas.
Lemma 3.4. Suppose \( \Omega \subset \mathbb{C}^n \) is a bounded strongly pseudoconvex domain with smooth boundary and \( \Phi \) is a biholomorphic map on a neighborhood of \( \overline{\Omega} \), \( \Phi(\Omega) = \Omega' \). Let \( X \) and \( X' \) be the functions defined in (2.1) for \( \Omega \) and \( \Omega' \) respectively. Then there exists \( \delta > 0 \) such that for \( (z, w) \in K_\delta \),

\[
|X(z, w)| \approx |X'(\Phi(z), \Phi(w))|.
\]

Proof. Fix a defining function \( r(z) \), then \( r \circ \Phi^{-1} \) is a defining function for \( \Omega' \). By Lemma 2.5 and Lemma 2.2, there exists \( \delta > 0 \) so that for \( (z, w) \in K_\delta \),

\[
|X(z, w)| \approx \delta(z) + \delta(w) + |z - w|^2 + \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j} (w_j - z_j) \right|.
\]

Similarly,

\[
|X'(\Phi(z), \Phi(w))| \approx \delta(\Phi(z)) + \delta(\Phi(w)) + |\Phi(z) - \Phi(w)|^2 + \left| \sum_{j=1}^{n} \frac{\partial r \circ \Phi^{-1}(\Phi(z))}{\partial z_j} (\Phi_j(w) - \Phi_j(z)) \right|
\]

The first three parts for \( X \): \( \delta(z) \), \( \delta(w) \) and \( |z - w|^2 \) are each equivalent to the corresponding ones for \( X' \) since \( \Phi \) preserves distances up to a constant. That is, both \( \Phi \) and \( \Phi^{-1} \) are Lipschitz. We look at the last one. Let \( \Phi \) be as in the assumption. Then \( r \circ \Phi^{-1} \) is a defining function for \( \Phi(\Omega) \). By Lemma 2.9 we only need to prove the result using this defining function. Now

\[
\sum_{j=1}^{n} \frac{\partial r \circ \Phi^{-1}(\Phi(z))}{\partial z_j} (\Phi_j(w) - \Phi_j(z))
\]

\[
= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} \frac{\partial \Phi^{-1}_j(\Phi(z))}{\partial z_j} \left( \sum_{k=1}^{n} \frac{\partial \Phi_j(z)}{\partial z_k} (w_k - z_k) + O(|w - z|^2) \right)
\]

\[
= \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \frac{\partial \Phi^{-1}_j(\Phi(z))}{\partial z_j} \frac{\partial \Phi_j(z)}{\partial z_k} \right) (w_k - z_k) + O(|w - z|^2)
\]

\[
= \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} \sum_{k=1}^{n} \delta_{ik} (w_k - z_k) + O(|w - z|^2)
\]

\[
= \sum_{i=1}^{n} \frac{\partial r(z)}{\partial z_i} (w_i - z_i) + O(|w - z|^2).
\]

From this it is clear that \( |X(z, w)| \approx |X'(\Phi(z), \Phi(w))| \). This completes the proof. \( \square \)
Roughly speaking, our approach to Theorem 3.1 is to prove a slightly stronger result about logarithms of absolute values of the functions and then apply the Jensen’s inequality. This will allow us to separate the part involving the function \( f \) in the theorem and concentrate on estimations about \( h \). As a first step, we consider the case when our domain is just the unit disc \( \mathbb{D} \) in \( \mathbb{C} \).

Lemma 3.5. There exists a constant \( C > 0 \), such that for any \( 0 < c < 1 \), \( z, w \in \mathbb{D} \) and \( a \in \mathbb{C} \), we have

\[
\log \frac{|z - a|}{|w - z| + \delta(w)} \leq \frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| d\nu_1(\lambda) - \log(\delta(w)) - \log c + C.
\]

Proof. Note that since \( 0 < c < 1 \), \( \delta(w) = 1 - |w| \), the disc \( \Delta(w, c\delta(w)) \) is contained in the unit disk \( \mathbb{D} \). We split the proof into two cases.

Case 1: \( a \notin \Delta(w, c\delta(w)) \). Then the function \( \log |\lambda - a| \) is harmonic in the disc \( \Delta(w, c\delta(w)) \). Therefore

\[
\frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| d\nu_1(\lambda) = \log |w - a|.
\]

For \( z \in \mathbb{D} \),

\[
\log \frac{|z - a|}{|w - z| + \delta(w)} = \frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| d\nu_1(\lambda) + \log \delta(w) - \log |w - a| + \log(\delta(w))
\]

\[
= \log \frac{|z - a|}{|w - z| + \delta(w)} - \log |w - a| + \log \delta(w)
\]

\[
= \log \frac{|z - a|\delta(w)}{|w - z| + \delta(w)|w - a|}.
\]

Let \( m, M \) be the minimal and maximal of the two numbers \(|w - z| + \delta(w)\) and \(|w - a|\). Since \(|z - a| \leq |z - w| + |w - a| \leq M + m\), we have \( M \geq 1/2|z - a| \). Also, since \(|w - a| \geq c\delta(w)\) by our assumption and \(|z - w| + \delta(w) \geq \delta(w) > c\delta(w)\), we have \( m \geq c\delta(w) \). Therefore

\[
\frac{|z - a|\delta(w)}{|w - z| + \delta(w)|w - a|} \leq 2/c.
\]

Hence

\[
\log \frac{|z - a|}{|w - z| + \delta(w)} = \frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| d\nu_1(\lambda) + \log \delta(w)
\]

\[
\leq \log 2 - \log c.
\]
This completes the proof for case 1.

Case 2: \( a \in \Delta(w, c\delta(w)) \). First, we make a change of variable. It is easy to verify that
\[
\frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| dv_1(\lambda)
\]
\[
= \frac{1}{\pi} \int_D \log \left| \eta - \frac{a - w}{c\delta(w)} \right| dv_1(\eta) + \log \delta(w) + \log c.
\]

In general, for \( a \in \mathbb{D} \),
\[
\int_D \log |\eta - a| dv_1(\eta)
\]
\[
= \int_0^1 \int_0^{2\pi} r \log |re^{i\theta} - a| d\theta dr
\]
\[
= \int_0^{|a|} \int_0^{2\pi} r \log |re^{i\theta} - a| d\theta dr + \int_{|a|}^1 \int_0^{2\pi} r \log |re^{i\theta} - a| d\theta dr
\]
\[
= \pi |a|^2 \log |a| + \int_{|a|}^1 2\pi r \log r dr
\]
\[
= \frac{\pi}{2} (|a|^2 - 1)
\]
\[
\geq -\frac{\pi}{2}.
\]

Therefore
\[
\frac{1}{\pi} \int_D \log |\eta - a| dv_1(\eta) \geq -1/2.
\]

So
\[
\frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| dv_1(\lambda) \geq \log \delta(w) + \log c - 1/2.
\]

On the other hand,
\[
\log \frac{|z - a|}{|w - z| + \delta(w)} \leq \log \frac{|z - a|}{|w - z| + |w - a|} \leq 0.
\]

So
\[
\log \frac{|z - a|}{|w - z| + \delta(w)} \leq \frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |\lambda - a| dv_1(\lambda)
\]
\[
- \log \delta(w) - \log c + 1/2.
\]

Taking \( C = \log 2 + 1/2 \) will complete the proof. \( \square \)
Lemma 3.6. There exists $C > 0$ such that for any polynomial $p \in \mathbb{C}[z]$ of degree $d$, $0 < c < 1$ and any $z, w \in \mathbb{D}$, we have
\[
\log \frac{|p(z)|}{(|z - w| + \delta(w))^d} \leq \frac{1}{v(\Delta(w, c\delta(w)))} \int_{\Delta(w, c\delta(w))} \log |p(\lambda)| d\lambda - d \log \delta(w) - d \log c + dC.
\]

Proof. The proof is immediate once we write $p(z) = a_0(z - a_1) \cdots (z - a_d)$ and apply Lemma 3.5. □

Remark 3.7. In general, if a polynomial $p$ of degree $d$ is defined on a disc $\Delta(\alpha, r)$, then the polynomial
\[
f(z) = p(rz + \alpha)
\]
has the same degree with $p$ and is defined on $\mathbb{D}$. For $z \in \Delta(\alpha, r)$ and $\Delta(w, s) \subseteq \Delta(\alpha, r)$, $\frac{w}{r} \in \mathbb{D}$, $\Delta\left(\frac{w}{r}, s/r\right) \subseteq \mathbb{D}$. Apply Lemma 3.6 to $f$, we get
\[
\int_{\Delta\left(\frac{w}{r}, s/r\right)} \log |f(\lambda)| d\lambda - d \log s/r + dC.
\]

We will use Lemma 3.6 in this form in the proof of Theorem 3.1.

Note also that Lemma 3.6 holds trivially for $p \equiv 0$ with any positive number $d$ since the left side is $-\infty$.

Lemma 3.8. Suppose $h \in \text{Hol}(U)$, where $U$ is an open neighborhood of $0 \in \mathbb{C}^n$, $h \not\equiv 0$. Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, such that $h$ is not identically 0 on each complex line $\mathbb{C} e_i \cap U$.

Lemma 3.8 can be implied by [3, Lemma 2, Page 33]. To avoid employing more terminologies we give a straightforward proof. We thank Dr. Dan for suggesting this proof to us.

Proof. In the case when $n = 1$, the conclusion is obvious. In the case when $n = 2$, notice that $\langle (z_1, z_2), (\bar{z}_2, -\bar{z}_1) \rangle = 0$ for all pairs $(z_1, z_2)$. Let
\[
f(z_1, z_2) = h(z_1, z_2) \bar{h}(\bar{z}_2, -\bar{z}_1).
\]
Then \( f \) is a product of two non-zero holomorphic functions. Thus \( f \) is not identically 0. Pick any \((z_1, z_2) \neq 0\) so that \( f(z) \neq 0\) and normalize \(\{(z_1, z_2), (\bar{z}_2, -\bar{z}_1)\}\) into an orthonormal basis. This will satisfy our condition.

Then we prove the general case by induction, suppose we have proved the result for \( U \subset \mathbb{C}^{n-1} \). Now for \( U \subset \mathbb{C}^n \), pick \( z \neq 0 \) so that \( h(z) \neq 0 \). Pick a two dimensional subspace \( L \subset \mathbb{C}^n \) containing \( z \), then \( h|_L \neq 0 \). Since \( \dim L = 2 \), by the previous argument we have orthonormal \( v_1 \) and \( v_2 \in L \) so that \( h \) is not identically 0 on \( C\Omega_1 \) and \( C\Omega_2 \). Now consider \( L' = v_1^+ \), since \( v_2 \in L' \), \( h|_{L'} \neq 0 \). By induction, we have orthonormal \( \{e_2, \ldots, e_n\} \subset L' \) such that \( h \) is not identically 0 on \( C\Omega_i \), \( i = 2, \ldots, n \). The set \( \{v_1, e_2, \ldots, e_n\} \) is the desired basis. This completes the proof. 

\[ \square \]

**Notations:** Under the setting of Theorem 3.1 and assume further that \( h \) is not identically 0 on the complex \( n-1 \) dimensional affine space passing \( \zeta \) and tangent to \( \Omega \) at \( \zeta \). Applying Lemma 3.8 on this \( n-1 \) dimensional affine space we get \( n \) - 1 vectors \( \{e_1^{\zeta}, \ldots, e_{n-1}^{\zeta}\} \) such that together with the unit normal vector at \( \zeta \), they form an orthonormal basis for \( \mathbb{C}^n \), and that \( h \) is not identically 0 on each complex line \( \zeta + \mathbb{C}e_i^{\zeta}, i = 1, \ldots, n-1 \). We denote \( e_n^{\zeta} \) to be the unit normal vector at \( \zeta \).

Now for any \( w \) in a sufficiently small neighborhood of \( \zeta \), let \( e_n^w \) be the unit normal vector at \( w \). Then \( e_n^w \) depends continuously on \( w \) and the definition is consistent at the point \( \zeta \). Fix \( e_n^\xi \), use the Gram-Schmidt method on \( \{e_1^{\zeta}, \ldots, e_{n-1}^{\zeta}\} \) to obtain a new orthonormal basis, denoted by \( \{e_1^w, \ldots, e_n^w\} \). For any \( n \) - tuple of complex numbers \( \xi = (\xi_1, \ldots, \xi_n) \), use \( \xi_w \) to denote the point in \( \mathbb{C}^n \) having coordinate \( \xi \) under the basis \( \{e_1^w, \ldots, e_n^w\} \), i.e., \( \xi_w = \sum_{i=1}^n \xi_i e_i^w \).

In the case when \( h \) is identically 0 on the \( n-1 \) dimensional affine space at \( \zeta \) tangent to \( \Omega \) at \( \zeta \), we can subtract a polynomial out of \( h \), and the rest is not identically 0 on the affine space. Indeed, assume for the moment that the normal vector at \( \zeta \) is \((0, \ldots, 0, 1)\), and that \( \zeta = 0 \), then \( h(\zeta) = z_n^m h'(\zeta) \) for some positive integer \( m \), in a neighborhood of \( \zeta \), where \( h' \) satisfies our assumption.

**Lemma 3.9.** Under the assumptions of Theorem 3.7 and assume further that \( h \) is not identically 0 along the normal direction at \( \zeta \), there exists a neighborhood \( \mathcal{V} \) of \( \zeta \) and constants \( \delta > 0 \), \( 0 < m < M \) and \( k > 0 \) such that for any \( i = 1, \ldots, n \) and any \( w \in \mathcal{V}, \delta(w) < \delta^2 \). Also, whenever \( \|\xi_w - w\| < \delta \), we have decompositions

\[ h(\xi_w) = W^{\zeta}_i(\xi)\varphi_i^w(\xi). \]

Here the functions \( W^\zeta_i \) and \( \varphi_i^\zeta \) are such that when we fix all variables but \( \xi_i \), the function \( W^\zeta_i \) is either identically 0 or a polynomial in \( \xi_i \) of degree less than \( k \). And \( m < |\varphi_i^\zeta(\xi)| < M \).
We write \( d(w, i, \xi) \) for the degree of the polynomial \( W^w_1(\xi) \). If the polynomial is identically 0, we let its degree be 0. Then \( d(w, i, \xi) \leq k \). When no confusion is caused, we simply write \( d \) for \( d(w, i, \xi) \).

Proof: First, notice that the condition \( \delta(w) < \delta^2 \) can be easily satisfied by simply shrinking the neighborhood \( V \). We show the rest can also be achieved.

According to our discussion proceeding Lemma 3.9, \( h \) can be written as a product of a polynomial with some \( h' \) such that \( h' \) is not identically 0 on the complex \( n - 1 \) dimensional affine space tangent to \( \Omega \) at \( \zeta \). So we get a set of parameterized basis \( \{ e^w_i \}_{i=1}^n \) where \( w \) ranges over a small neighborhood \( V_1 \) of \( \zeta \).

We prove the lemma for \( h' \), and the result for \( h \) follows immediately. For simplicity, write \( h \) for \( h' \). The case when \( h(\zeta) \neq 0 \) is obvious, we assume \( h(\zeta) = 0 \).

By our construction, \( h \) is not identically 0 on each complex line \( \zeta + C \Omega \). The proof for all \( i \)'s are the same. For convenience, we only prove the case \( i = 1 \).

Without loss of generality, assume \( \zeta = 0 \). Since zero points in dimension 1 is isolated, we can take \( r > 0 \) small enough so that the function \( h((z_1, 0')_\zeta) \) has no zero points other than \( z_1 = 0 \) on the closed disc \( \{ (z_1, 0')_\zeta : |z_1| \leq r \} \). Denote \( m_1 \) for its degree. By continuity, there exists \( \epsilon > 0 \) such that whenever \( |(\xi_2, \ldots, \xi_n)| < \epsilon \) and \( |w - \xi| < \epsilon \), \( h((z_1, \xi_2, \ldots, \xi_n)_w) \) has no zeros on the closed ring \( \{(z_1, \xi_2, \ldots, \xi_n)_w : r/2 \leq |z_1| \leq r \} \). By Rouché’s Theorem, the function has exactly \( m_1 \) zeros (counting multiplicity) in the disc \( \{(z_1, \xi_2, \ldots, \xi_n)_w : |z_1| < r/2 \} \).

Therefore, for such \( \xi' := (\xi_2, \ldots, \xi_n) \) and \( w \) we have decomposition

\[
h((z_1, \xi')_w) = W^w_1(z_1, \xi') \varphi^w_1(z_1, \xi').
\]

Here \( W^w_1 \) is a monic polynomial of degree \( m_1 \) in \( z_1 \), with zeros inside \( \{ z_1 : |z_1| < r/2 \} \) and \( \varphi^w_1 \) is holomorphic in \( z_1 \) and zero-free on \( \{ z_1 : |z_1| \leq r \} \). In fact, for \( |z_1| = r \),

\[
|\varphi^w_1(z_1, \xi')| = \frac{|h((z_1, \xi')_w)|}{|W^w_1(z_1, \xi')|}.
\]

Since

\[
\left(\frac{1}{r}\right)^{m_1} \leq |W^w_1(z_1, \xi')| \leq \left(\frac{2}{r}\right)^{m_1}
\]

and \( |h((z_1, \xi')_w)| \) can be taken uniformly bounded and bounded away from 0 for all \( \xi' \) and \( w \), by possibly shrinking \( V_1 \) and \( \epsilon \). So there exists \( 0 < m < M \) such that \( m \leq |\varphi^w_1(z_1, \xi')| \leq M \) on the circle \( \{ z_1 : |z_1| = r \} \). By the Maximum Principle, it also holds for \( |z_1| < r \).

Shrink \( \epsilon \) to make \( \epsilon < r \). Now for \( |\xi_1| < r, |\xi'| < \epsilon \) and \( |w - \xi| < \epsilon \), we have the above decomposition and \( \varphi^w_1 \) has the above estimation. Now take \( V = B(\xi, \epsilon/2) \) and \( \delta = \epsilon/2 \), if \( |\xi_w - w| < \delta \) and \( w \in V \),

\[
|\xi| = |\xi_w| \leq |\xi_w - w| + |w| < \epsilon < r.
\]
This completes our proof when \( h = h' \) and \( i = 1 \). For the general case, one easily sees that by modifying the constants the decomposition works for all \( i \). We remind the reader that from \( h' \) to \( h \), when we multiply a polynomial to \( W''_m \), the resulting polynomial might be zero on certain complex lines. But this will not influence our final estimation. \( \square \)

**Proof of Theorem 3.7** First, we could replace \( U \) by a smaller neighborhood so that \((z, w) \in K_\delta \) for any \( z, w \in U \cap \Omega \) for some \( \delta > 0 \) so that the all the previous lemmas involving \( K_\delta \) holds.

Note that \( v(E(w, 1)) \approx \mu(w) \approx |r(w)\mu| \), it is sufficient to show that

\[
|h(z)f(w)| \leq \frac{|X(z, w)|^N}{|r(w)|^{N}} \int_{E(w, 1)} |h(\lambda)f(\lambda)|dv_n(\lambda).
\]

First, if \( h(\gamma) \neq 0 \). Since \( |X(z, w)| \approx \delta(w) \approx |r(w)| \), we only need to show

\[
|h(z)f(w)| \leq \frac{1}{v(E(w, 1))} \int_{E(w, 1)} |h(\lambda)f(\lambda)|dv_n(\lambda).
\]

Take a neighborhood \( V_1 \subseteq U \) of \( \zeta \) so that \( 0 < m < |h| < M \) on \( V_1 \), for some constant \( m, M \). Take \( V \subseteq V_1 \) and \( \delta > 0 \) so that \( B(w, \delta) \subseteq V \) for \( w \in V \). By Lemma 2.4, the size of \( E(w, 1) \) tends to 0 as \( w \) approaches \( \partial \Omega \), so we can shrink \( V \) so that \( E(w, 1) \subseteq V \) whenever \( w \in V \cap \Omega \). Then for \( w \in V \cap \Omega, z \in B(w, \delta) \cap \Omega \),

\[
\begin{align*}
|f(w)| & \leq \frac{1}{v(P_w(a_1\delta(w)^{1/2}, b_1\delta(w)))} \int_{P_w(a_1\delta(w)^{1/2}, b_1\delta(w))} |f(\lambda)|dv_n(\lambda) \\
& \leq \frac{1}{v(E(w, 1))} \frac{1}{v(P_w(a_1\delta(w)^{1/2}, b_1\delta(w)))} v(E(w, 1)) \int_{E(w, 1)} |f(\lambda)|dv_n(\lambda) \\
& \leq \frac{1}{v(E(w, 1))} \int_{E(w, 1)} |f(\lambda)|dv_n(\lambda).
\end{align*}
\]

Since \( 0 < m < |h(z)| < M \) for \( z \in V_1 \), we have

\[
|h(z)f(w)| \leq M|f(w)| \leq \frac{M}{v(E(w, 1))} \int_{E(w, 1)} |f(\lambda)|dv_n(\lambda) \leq \frac{M}{mv(E(w, 1))} \int_{E(w, 1)} |h(\lambda)f(\lambda)|dv_n(\lambda) \leq \frac{1}{v(E(w, 1))} \int_{E(w, 1)} |h(\lambda)f(\lambda)|dv_n(\lambda).
\]

This completes the proof for the case \( h(\gamma) \neq 0 \).
Now assume $h(\zeta) = 0$. First, we show that we could assume $h$ to be not identically 0 along the normal direction at $\zeta$.

**Claim:** There is a biholomorphic map $\Phi$ defined on a neighborhood of $\overline{\Omega}$ such that $h \circ \Phi$ is not identically 0 along the complex normal direction of $\Phi(\Omega)$ at the point $\Phi(\zeta)$.

Assume the claim and suppose we have proved the theorem in the case when $h$ is not identically 0 along the complex normal direction. Then the result holds for the function $h' = h \circ \Phi^{-1}$ defined in a neighborhood $U' = \Phi(U)$ of $\zeta' = \Phi(\zeta)$, for the domain $\Omega' = \Phi(\Omega)$. Then we have $V' \subseteq U'$, $\delta' > 0$ and $N > 0$ as stated in the theorem. Let $V = \Phi^{-1}(V')$, then we can find $\delta > 0$ so that $\Phi(B(w, \delta)) \subseteq B(\Phi(w), \delta')$ for any $w \in V$. For $f \in \text{Hol}(E(w, 1))$, since biholomorphic maps preserve the Kobayashi distance, $f' = f \circ \Phi^{-1} \in \text{Hol}(E(\Phi(w), 1))$. So for any $w \in V \cap \Omega$ and $z \in B(w, \delta) \cap \Omega$ we have

$$|h'(\Phi(z))f'(\Phi(w))| \leq \frac{|X'(\Phi(z), \Phi(w))|^N}{|r \circ \Phi^{-1}(\Phi(w))|^{N+1}} \int_{E(\Phi(w), 1)} |h'(\lambda)f'(\lambda)|dv_{n}(\lambda).$$

Since $\Phi$ is biholomorphic in a neighborhood of $\overline{\Omega}$, the absolute value of its real Jacobian is both bounded above and away from 0. Combining this with Lemma \[3,4] we get

$$|h(z)f(w)| = |h'(\Phi(z))f'(\Phi(w))| \leq \frac{|X(z, w)|^N}{|r(w)|^{N+1}} \int_{E(w, 1)} |h(\lambda)f(\lambda)|dv_{n}(\lambda).$$

This is our desired result.

Now we prove the claim. For any $r > 0$ one can take a ball $B$ in $\mathbb{C}^n$ of radius $r$ that is tangent to $\Omega$ at the point $\zeta$. If we make $r$ small enough we can also assume that the center of $B$ is contained in $\Omega \cap U$. By doing a translation and an invertible linear transformation (which are biholomorphic maps) we can assume that $B$ is the unit ball in $\mathbb{C}^n$. Now $0 \in \Omega \cap U$, so $h$ is defined in a neighborhood of 0. Since $h$ is not identically 0, it is not identically 0 in any open set. Since $\Omega$ is bounded, we can find a $\alpha$ close enough to 0 so that $h(\alpha) \neq 0$ and the automorphism of $\mathbb{B}_n$ defined by

$$\varphi_{\alpha}(z) = \frac{\alpha - P_{\alpha}(z) - (1 - |\alpha|^2)^{1/2}Q_{\alpha}(z)}{1 - \langle z, \alpha \rangle}$$

is defined and biholomorphic in a neighborhood of $\overline{\Omega}$. The map $\varphi_{\alpha}$ has properties $\varphi_{\alpha}(0) = \alpha$ and $\varphi_{\alpha}^2 = \text{id}$ (cf. \[23\]). It is easy to show that the domains $\varphi_{\alpha}(\Omega)$ and $\varphi_{\alpha}(\mathbb{B}_n) = \mathbb{B}_n$ is tangent at $\varphi_{\alpha}(\zeta)$. Therefore they have the same complex normal direction at $\varphi_{\alpha}(\zeta)$, which is just the one determined by the points 0 and $\varphi_{\alpha}(\zeta)$. Since $h \circ \varphi_{\alpha}^{-1}(0) = h(\alpha) \neq 0$, $h \circ \varphi_{\alpha}$ is not identically 0 along the complex normal direction. This proves the claim.
Now we prove the theorem assuming \( h(\zeta) = 0 \) and \( h \) is not identically 0 along the complex normal direction. At this point, we could apply Lemma \ref{3.9} to get decompositions

\[
h(\xi_w) = W^w(\xi) \varphi^w(\xi)
\]

with stated properties.

By Lemma \ref{2.4} there exists \( a_1, a_2, b_1, b_2 > 0 \) for \( w \) close enough to \( \partial \Omega \), such that

\[
P_w(a_1 \delta(w), b_1 \delta(w)^{1/2}) \subseteq E(w, 1) \subseteq P_w(a_2 \delta(w), b_2 \delta(w)^{1/2}).
\]

Since these sets have comparable volume measures, we only need to prove Theorem \ref{3.1} with \( E(w, 1) \) replaced by the polydisk above on the left.

We will use induction to prove the Lemma.

Let \( V, \delta, k, m, M \) as in Lemma \ref{3.9}. Let \( a = \min\{a_1, b_1, 1\} \). Fix \( w \in V \cap \Omega \), in the rest of the proof, we will always use the orthonormal basis \( \{e_i^w\} \) instead of the canonical one. To simplify notation, we omit any \( w \) in the subscript or superscript. Therefore \( \xi \) means \( \xi_w \) and \( W \) means \( W^w \), etc. We could also do a translation to make \( w = 0 \).

For \( z \in B(w, \frac{\delta}{4n}) \), suppose \( z = (z_1, \ldots, z_n) \). Since \( |z - w| < \delta \), the polynomial in \( \lambda \), \( W_1(\lambda, z_2, \ldots, z_n) \) is well defined. Since \( |z_1| \leq |z| = |z - w| \), the point \( z_1 \) is in the disc \( \Delta(0, |z - w| + \delta(w)^{1/2}) \). Also, \( \Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}) \subseteq \Delta(0, |z - w| + \delta(w)^{1/2}) \). By Lemma \ref{3.6} and the Remark after it,

\[
\log \frac{|W_1(z_1, \ldots, z_n)|}{(|z_1|/(|z - w| + \delta(w)^{1/2}) + 1)^d} \leq \frac{1}{v(\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}))} \int_{\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2})} \log |W_1(\lambda_1, z_2, \ldots, z_n)|d\nu_1(\lambda_1)

- d \log \frac{\delta(w)}{|z - w| + \delta(w)^{1/2}} + dC.
\]

Since the denominator on the left side is greater than 1 and since \( |z - w| + \delta(w)^{1/2} \leq |X(z, w)|^{1/2} \), by changing the constant \( C \) we have

\[
\log |W_1(z_1, \ldots, z_n)| \leq \frac{1}{v(\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}))} \int_{\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2})} \log |W_1(\lambda_1, z_2, \ldots, z_n)|d\nu_1(\lambda_1)

- d/2 \log \frac{\delta(w)}{|X(z, w)|} + dC.
\]

Now for \( \lambda_1 \in \Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}) \),

\[
|\lambda_1, z_2, \ldots, z_n) - w| \leq |\lambda_1| + |z - w| \leq \frac{\delta(w)^{1/2}}{4n} + \frac{\delta}{4n} < \frac{\delta}{2n} \leq \delta
\]
Therefore

\[ \varphi_1(\lambda_1, z_2, \ldots, z_n) \] is well defined and bounded below and above by \( 0 < m < M \). Therefore

\[
\begin{align*}
\log |h(z_1, \ldots, z_n)| &= \log |W_1(z_1, \ldots, z_n)| + \log |\varphi(z_1, \ldots, z_n)| \\
&\leq \frac{1}{v(\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}))} \int_{\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2})} \log |W_1(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
&\quad - d/2 \log \frac{\delta(w)}{|X(z, w)|} + dC + \log M \\
&\leq \frac{1}{v(\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}))} \int_{\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2})} \log |h(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
&\quad - d/2 \log \frac{\delta(w)}{|X(z, w)|} + dC + \log M \frac{m}{m} \\
&\leq \frac{1}{v(\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}))} \int_{\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2})} \log |h(\lambda_1, z_2, \ldots, z_n)| dv_1(\lambda_1) \\
&\quad - k/2 \log \frac{\delta(w)}{|X(z, w)|} + kC + \log M \frac{m}{m}.
\end{align*}
\]

Here the last inequality is because \( d = d(w, 1, z) \leq k \). Since \( \frac{|X(z, w)|}{\delta(w)} \geq 1, \frac{M}{m} \geq 1 \), therefore by enlarging the constant \( C \) we could make the sum of the coefficients of \( d \) positive. Since \( d \leq k \), we have the last inequality.

Now, for \( \lambda_1 \in \Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}) \), we have shown that \( |(\lambda_1, z_2, \ldots, z_n) - w| < \frac{\delta}{2n} < \delta \), also notice that for \( \lambda_2 \in \Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}) \),

\[ |(\lambda_1, \lambda_2, z_3, \ldots, z_n) - w| \leq |\lambda_1| + |\lambda_2| + |z - w| \leq \frac{2\delta(w)^{1/2}}{4n} + \frac{\delta}{4n} \leq \frac{3\delta}{4n} < \delta, \]

This means that we could replace \( z \) by \((\lambda_1, z_2, \ldots, z_n)\) and repeat the above argument on the second index. We get

\[
\begin{align*}
\log |h(\lambda_1, z_2, \ldots, z_n)| &\leq \frac{1}{v(\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}))} \int_{\Delta(0, \frac{\delta}{4n} \delta(w)^{1/2})} \log |h(\lambda_1, \lambda_2, z_3, \ldots, z_n)| dv_1(\lambda_2) \\
&\quad - k/2 \log \frac{\delta(w)}{|X(z, w)|} + kC + \log M \frac{m}{m}.
\end{align*}
\]

In general, for \( \lambda_i \in \Delta(0, \frac{\delta}{4n} \delta(w)^{1/2}), i = 1, \ldots, n - 1 \)

\[ |(\lambda_1, \ldots, \lambda_i, z_{i+1}, \ldots, z_n) - w| < \frac{(i + 1)\delta}{4n} < \delta/4, \]
So we can repeat the arguments above for each of the first \( n - 1 \) indices to get
\[
\log |h(\lambda_1, \ldots, \lambda_{i}, z_{i+1}, \ldots, z_{n})| \\
\leq \frac{1}{v(\Delta(0, \frac{a}{4n} \delta(w) \sqrt{M}))} \int_{\Delta(0, \frac{a}{4n} \delta(w) \sqrt{M})} \log |h(\lambda_1, \ldots, \lambda_{i+1}, z_{i+2}, \ldots, z_{n})| dv_1(\lambda_{i+1}) \\
- \frac{k}{2} \log \frac{\delta(w)}{|X(z, w)|} + kC + \log \frac{M}{m}.
\]
Combining the inequalities in each step, we get
\[
\log |h(z_1, \ldots, z_n)| \\
\leq \frac{1}{v(\Delta(0, \frac{a}{4n} \delta(w) \sqrt{M}))} \int_{\Delta(0, \frac{a}{4n} \delta(w) \sqrt{M})} \log |h(\lambda', z_n)| dv_{n-1}(\lambda') \\
- \frac{k(n-1)}{2} \log \frac{\delta(w)}{|X(z, w)|} + k(n-1)C + (n-1) \log \frac{M}{m}.
\]
The \( n-\text{th} \) index represents the normal direction at \( w \), we handled it a little differently.

We have already showed that \(|(\lambda', z_n) - w| < \frac{\delta}{4} \). So the decomposition in Lemma 3.9 still makes sense. For the polynomial \( W_n(\lambda', A_n) \), apply Lemma 3.6 on the Disc \( \Delta(1, 1 + |z_n| + \delta(w)) \), taking average on \( \Delta(0, \frac{a}{4} \delta(w)) \). Clearly \( z_n \in \Delta(1, 1 + |z_n| + \delta(w)) \).

We get
\[
\log |W_n(\lambda', z_n)| \\
\leq \frac{1}{v(\Delta(0, \frac{a}{4} \delta(w)))} \int_{\Delta(0, \frac{a}{4} \delta(w))} \log |W_n(\lambda', A_n)| dv_1(A_n) - d \log (a/4) \delta(w) + dC
\]
\[
= \frac{1}{v(\Delta(0, \frac{a}{4} \delta(w)))} \int_{\Delta(0, \frac{a}{4} \delta(w))} \log |W_n(\lambda', A_n)| dv_1(A_n) - d \log \delta(w) + dC.
\]

Note that the constant \( C \) has enlarged in the process.

Again, since \( 2|z_n| + \delta(w) \leq |X(z, w)| \), we have
\[
\log |W_n(\lambda', z_n)| \\
\leq \frac{1}{v(\Delta(0, \frac{a}{4} \delta(w)))} \int_{\Delta(0, \frac{a}{4} \delta(w))} \log |W_n(\lambda', A_n)| dv_1(A_n) \\
- d \log \delta(w) + d \log |X(z, w)| + dC
\]
\[
\leq \frac{1}{v(\Delta(0, \frac{a}{4} \delta(w)))} \int_{\Delta(0, \frac{a}{4} \delta(w))} \log |W_n(\lambda', A_n)| dv_1(A_n) \\
+ k \log \frac{|X(z, w)|}{\delta(w)} + kC.
\]

21
And therefore

\[ \log |h(\lambda', z_n)| \leq \frac{1}{v(\Delta(0, \frac{1}{4}\delta(w)))} \int_{\Delta(0, \frac{1}{4}\delta(w))} \log |h(\lambda', \lambda_n)|dv_1(\lambda_n) \]

\[ + k \log \frac{|X(z, w)|}{\delta(w)} + kC + \log \frac{M}{m}. \]

Again, substituting it into the previous estimation we get

\[ \log |h(z)| \leq \frac{1}{v(P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)))} \int_{P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w))} \log |h(\lambda)|dv_1(\lambda) \]

\[ + (k + \frac{k(n - 1)}{2}) \log \frac{|X(z, w)|}{\delta(w)} + knC + n \log \frac{M}{m}. \]

Note that \( \Delta(0, \frac{a}{4n}\delta(w)^{1/2})^{n-1} \times \Delta(0, \frac{a}{4}\delta(w)) = P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)). \) Combining the constants, we get

\[ \log |h(z)| \leq \frac{1}{v(P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)))} \int_{P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w))} \log |h(\lambda)|dv_1(\lambda). \] (3.1)

\[ + N \log \frac{|X(z, w)|}{\delta(w)} + C. \] (3.2)

Here \( N = (k + \frac{k(n - 1)}{2}). \) The polydisc \( P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)) \) is obviously contained in the polydisc \( P_w(a_1\delta(w)^{1/2}, b_1\delta(w)). \) Since \( \log |f| \) is pluri-subharmonic, we have

\[ \log |f(w)| \leq \frac{1}{v(P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)))} \int_{P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w))} \log |f(\lambda)|dv_1(\lambda). \]

Adding them up we have

\[ \log |h(z)f(w)| \leq \frac{1}{v(P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)))} \int_{P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w))} \log |h(\lambda)f(\lambda)|dv_1(\lambda) \]

\[ + N \log \frac{|X(z, w)|}{\delta(w)} + C. \]

Then apply the Jensen’s inequality, we get

\[ |h(z)f(w)| \leq \frac{|X(z, w)|^N}{\delta(w)^N} \frac{1}{v(P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w)))} \int_{P_w(\frac{a}{4n}\delta(w)^{1/2}, \frac{a}{4}\delta(w))} |h(\lambda)f(\lambda)|dv_1(\lambda). \]
Finally, since \( P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w)) \subseteq E(w, 1) \) and \( v(Q_w) = v(E(w, 1)) \approx \delta(w)^{r+1} \), we have for any \( w \in V \) and \( z \in B(w, \delta^r) \):

\[
|h(z)f(w)| \leq \frac{|X(z, w)|^N}{\delta(w)^{N+n+1}} \int_{E(w, 1)} |h(\lambda)| dv_w(\lambda)
\]

\[
\approx \frac{|X(z, w)|^N}{|\nu(w)|^{N+n+1}} \int_{E(w, 1)} |h(\lambda)| dv_w(\lambda).
\]

This completes the proof. \( \square \)

**Proof of Corollary 3.3** As usual, we will not keep track of the constant \( C \) in the estimation. So the notation \( C \) may denote different constants in the proof.

A key step in the proof of Theorem 3.1 is to obtain inequality 3.1. We could apply the proof of Lemma 3.1 to every point \( \zeta \in \partial \Omega \). Since \( \Omega \) is bounded, \( \partial \Omega \) is compact. Thus we get a finite cover \( \{ V_i \}_{i=1}^m \) of \( \partial \Omega \) where each \( V_i \) corresponds to some point \( \zeta_i \) in \( \partial \Omega \). It is easy to see that one can adjust so that the same set of constants work for all points. That is to say, there exist a neighborhood \( V = \bigcup V_i \) of \( \partial \Omega \) and constants \( \delta > 0, N > 0, C > 0 \) such that \( \forall w \in V, |z - w| < \delta \),

\[
\log |h(z)| \leq \frac{1}{v(P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w)))} \int_{P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w))} \log |h(\lambda)| dv_w(\lambda)
\]

\[
\quad + N \log \left( \frac{|X(z, w)|}{\delta(w)} \right) + C.
\]

Note that this include pairs \( (z, w) \in K_{\delta'} \) for some \( \delta' > 0 \). We are left with the case when \( (z, w) \notin K_{\delta'} \). For such pairs, \( F(z, w) \) is bounded below and above.

Fix finite number of points \( z_1, \ldots, z_k \in \Omega \) so that \( h(z_i) \neq 0 \) and for any \( w \in V \) there exists some \( z_i \) so that \( |z_i - w| < \delta \). Also, \( |X(z, w)| \) is bounded above for all \( z, w \in \Omega \). Therefore for any \( w \in V, \)

\[
\frac{1}{v(P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w)))} \int_{P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w))} \log |h(\lambda)| dv_w(\lambda)
\]

\[
\quad + N \log \frac{1}{\delta(w)} + C
\]

\[
\geq \frac{1}{v(P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w)))} \int_{P_w(\frac{\delta}{4n}\delta(w)^{1/2}, \frac{\delta}{4}\delta(w))} \log |h(\lambda)| dv_w(\lambda)
\]

\[
\quad + N \log \frac{|X(z, w)|}{\delta(w)} + C
\]

\[
\geq \log |h(z_i)| \geq C.
\]
By compactness, for \( w \in \Omega \setminus V \),
\[
\frac{1}{v(P_w(\frac{4}{4n}\delta(w)^{1/2}, \frac{4}{4n}\delta(w))))} \int_{P_w(\frac{4}{4n}\delta(w)^{1/2}, \frac{4}{4n}\delta(w))} \log |h(\lambda)|d\nu_\lambda(\lambda) + N \log \frac{1}{\delta(w)} > C
\]
for some constant \( C \). For \((z, w) \notin K_{\delta'}\), \( F(z, w) \) is bounded below. Thus
\[
\frac{1}{v(P_w(\frac{4}{4n}\delta(w)^{1/2}, \frac{4}{4n}\delta(w))))} \int_{P_w(\frac{4}{4n}\delta(w)^{1/2}, \frac{4}{4n}\delta(w))} \log |h(\lambda)|d\nu_\lambda(\lambda) + N \log \frac{F(z, w)}{\delta(w)} > C
\]
for some constant \( C \). Since \( h \) is bounded above on \( \Omega \), there is some constant \( C \) such that
\[
\log |h(z)| - C.
\]
Therefore the inequality above holds for all \( z, w \in \Omega \).

The rest of proof is as in the last part of Theorem 3.1. This completes the proof. \( \square \)

4 Main Result

**Theorem 4.1.** Suppose \( \Omega \subseteq \mathbb{C}^n \) is a bounded strongly pseudoconvex domain with smooth boundary, \( h \in Hol(\Omega) \), then the principal submodule of the Bergman module \( L^2_\alpha(\Omega) \) generated by \( h \) is \( p \)-essentially normal for all \( p > n \).

The proof of the following Lemma is the same as [8, Lemma 5.1].

**Lemma 4.2.** Suppose \( 2 \leq p < \infty \) and \( G(z, w) \) is a measurable in \( \Omega \times \Omega \). Let \( A_G \) be the integral operator on \( L^2(\Omega) \) defined by
\[
A_G f(z) = \int_{\Omega} \frac{G(z, w)}{F(z, w)^{p/2(n+1)}} f(w) d\nu_n(w).
\]
If
\[
\int_{\Omega} \int_{\Omega} \frac{|G(z, w)|^p}{F(z, w)^{p(n+1)/2(n+1)}} d\nu_n(z) d\nu_n(w) < \infty,
\]
then the operator \( A_G \) is in the Schatten \( p \) class \( S^p \).

**Lemma 4.3.** Let \( M_i^* \) be the adjoint of multiplication operator on the Bergman space \( L^2_\alpha(\Omega) \), \( i = 1, \ldots, n \). Let \( l \) be a positive integer and let \( G_l \) be the operator defined by
\[
G_l f(z) = \int_{\Omega} \bar{w}_l f(w) G_l(z, w) |r|^{l} d\nu_n(w).
\]
Then the operator $G_i$ is a bounded operator on $L^2_a(\Omega)$ and $G_i - M^*_z$ is in the Schatten $p$ class $S_p$ on $L^2_a(\Omega)$, for any $p > 2n$.

**Proof.** The fact that $G_i$ is bounded on $L^2_a(\Omega)$ can be obtained by Schur’s test. By Lemma 2.6,

$$|G_if(z)| \lesssim \int_{\Omega} \frac{|f(w)||r(w)|}{F(z,w)^{p+1+\frac{1}{2}}} dv_n(w).$$

Let $h(w) = |r(w)|^{-1/2}$, by Lemma 2.12 and Lemma 2.8,

$$\int_{\Omega} \frac{|r(w)|}{F(z,w)^{p+1+\frac{1}{2}}} h(w) dv_n(w) \lesssim h(z)$$

and

$$\int_{\Omega} \frac{|r(z)|}{F(z,w)^{p+1+\frac{1}{2}}} h(z) dv_n(w) \lesssim h(w).$$

By Schur’s test, $G_i$ defines a bounded operator on $L^2_a(\Omega)$.

Now for any $f \in L^2_a(\Omega) \subseteq L^2_{a,d}(\Omega)$,

$$(G_i - M^*_z)f(z)$$

$$= \int_{\Omega} \overline{\tilde{w}_i} f(w)(|r(w)| G_i(z,w) - K(z,w)) dv_n(w)$$

$$= \int_{\Omega} (\overline{\tilde{w}_i} - \overline{z_i}) f(w)(|r(w)| G_i(z,w) - K(z,w)) dv_n(w).$$

Since $|r(w)| \leq F(z,w)$, we have

$$\left| (G_i - M^*_z)f(z) \right| \lesssim \int_{\Omega} |w - z||f(w)| \frac{1}{F(z,w)^{p+1+\frac{1}{2}}} dv_n(w).$$

Now write $G(z,w) = |z - w|$ and apply Lemma 4.2 by Lemma 2.12 for any $2n < p < 2(n+1)$

$$\int_{\Omega} \int_{\Omega} \frac{|z-w|^p}{F(z,w)^{2(n+1)}} \leq \int_{\Omega} \int_{\Omega} \frac{1}{F(z,w)^{2n+2-p/2}} dv_n(w) dv_n(z)$$

$$\leq \int_{\Omega} \frac{|r(w)|^{p/2 - n-1}}{dv_n(w)} < \infty.$$ 

Therefore $G_i - M^*_z \in S^p$ for $2n < p < 2(n+1)$. If $p \geq 2(n+1)$, $S^{2n+1} \subseteq S^p$, we also have $G_i - M^*_z \in S^p$. This completes the proof. \[\square\]
Proof of Theorem 4.4. The fact that \( L^2_p(\Omega) \) itself is \( p \) essentially normal for \( p > n \) follows from

\[
(M_z^* M_z - M_z M_z^*) f(z) = \int_{\Omega} (|w_i|^2 - z_i \bar{w}_i) f(w) K(z, w) = \int_{\Omega} \bar{w}_i (w_i - z_i) f(w) K(z, w) = \int_{\Omega} \bar{w}_i (w_i - z_i) f(w) K(z, w) = \int_{\Omega} |w_i - z_i|^2 f(w) K(z, w)
\]

and a similar argument as in the proof of Lemma 4.3. By Proposition 4.1 in [1], we only need to show that the commutator

\[
[P, M_z] = PM_z - M_z P = PM_z - PM_z = PM_z P^g
\]

is in \( S^p \) for \( p > 2n \). Here \( P \) is the orthogonal projection onto the principal submodule generated by \( h \). This is equivalent to \( P^g M_z^* P \) being in the same class, which, by Lemma 4.3, is equivalent to \( P^g G_i P \) being in the same class. Functions of the form \( hf \) where \( f \in L^2_p(\Omega) \) is dense in the submodule generated by \( h \). Notice that

\[
||P^g G_i P(hf)|| \leq ||G_i(hf) - M_h G_i f||.
\]

We only need to estimate the norm on the right side. Using a similar trick as above, we get

\[
G_i(hf) - M_h G_i f(z) = \int_{\Omega} \bar{w}_i (h(w) - h(z)) f(w) G_i(z, w) |r(w)|^l dv_n(w) = \int_{\Omega} (\bar{w}_i - \bar{z}_i) (h(w) - h(z)) f(w) G_i(z, w) |r(w)|^l dv_n(w) = \int_{\Omega} (\bar{w}_i - \bar{z}_i) h(w) f(w) G_i(z, w) |r(w)|^l dv_n(w) - \int_{\Omega} (\bar{w}_i - \bar{z}_i) h(z) f(w) G_i(z, w) |r(w)|^l dv_n(w).
\]

So

\[
|G_i(hf) - M_h G_i f(z)| \leq \int_{\Omega} |h(w) f(w)| \frac{1}{F(z, w)^{n+1/2}} dv_n(w) + \int_{\Omega} |h(z) f(w)| \frac{|w - z||r(w)|^l}{F(z, w)^{n+1+l}} dv_n(w) = I(hf) + II(hf).
\]

26
We look at the second part. By Lemma 3.3

\[
II(hf)(z) \leq \int_{E(w,1)} |h(\lambda)f(\lambda)| \frac{F(z, w)^N}{|r(\lambda)|^{n+1+N}} \frac{|w-z|}{F(z, w)^{n+1+N}} dv_n(\lambda)dv_n(w)
\]

\[
\leq \int_{E(w,1)} |h(\lambda)f(\lambda)| \frac{|r(\lambda)|^{\frac{1}{2}-\frac{N}{n+1}}}{F(z, w)^{n+1/2+N}} dv_n(\lambda)dv_n(w)
\]

\[
= \int_{E(w,1)} \frac{|r(\lambda)|^{\frac{1}{2}-\frac{N}{n+1}}}{F(z, w)^{n+1/2+N}} dv_n(w)|h(\lambda)f(\lambda)|dv_n(\lambda)
\]

\[
\leq \int_{E(w,1)} \frac{|r(\lambda)|^{\frac{1}{2}-\frac{N}{n+1}}}{F(z, w)^{n+1/2+N}} |h(\lambda)f(\lambda)|dv_n(\lambda).
\]

where the last inequality comes from Lemma 2.11 and Lemma 2.4. We could take \( l > N \) in the beginning. Since \( |r(\lambda)| \leq F(z, \lambda) \), we get

\[
II(hf)(z) \leq \int_{E(w,1)} \frac{1}{F(z, \lambda)^{n+1/2}} |h(\lambda)f(\lambda)|dv_n(\lambda).
\]

Altogether we have

\[
|G_i(hf) - M_iG_i f(z)| \leq \int_{E(w,1)} |h(w)f(w)| \frac{1}{F(z, w)^{n+1/2}} dv_n(w).
\]

Take \( G(z, w) = F(z, w)^{1/2} \) and apply Lemma 4.2 as in the proof of Lemma 4.3 we get our desired result. This completes the proof. \( \square \)

5 Further Results on the Unit Ball

This section is dedicated to some further results on submodules of the Bergman module \( L^2_\Omega(B^n) \). Besides their main result on \( p \)-essential normality, the first author and K. Wang [7] also obtained a characterization of functions in the principal submodule \( [p] \in L^2_\Omega(B^n) \) generated by a polynomial \( p \). They also obtained some result on the essential spectrum of the module actions on \( [p] \). We show the same is true for any holomorphic function \( h \) defined in a neighborhood of \( \overline{B^n} \). In addition, we show that the Geometric Arveson-Douglas Conjecture is true for a pure analytic subset of codimension 1.

For convenience and future reference, we restate the two main results in the previous sections for the unit ball.

**Theorem 5.1.** Suppose \( h \) is a holomorphic function defined on a neighborhood of the closed unit ball \( \overline{B^n} \), then there exists a constant \( N \) such that for any function \( f \in Hol(B^n) \) and any \( z, w \in B^n \),

\[
|h(z)f(w)| \leq \frac{|1 - \langle w, z \rangle|^N}{(1 - |w|^2)^{N+1}} \int_{E(w,1)} |h(\lambda)f(\lambda)|dv_n(\lambda).
\]
Theorem 5.2. Suppose $h$ is a holomorphic function defined on a neighborhood of $\mathbb{B}_n$, then the principal submodule

$$[h] := \{ hf : f \in L^2_a(\mathbb{B}_n) \}$$

is $p$-essentially normal for $p > n$.

5.1 About Principal Submodules

For a generator $h$ as in Theorem 5.2, we are going to get a description of functions in the submodule $[h]$.

Let the measure $d\mu_h = |h|^2 dv_n$. Let $L^2(\mu_h)$ be the space of functions that are square integrable under this measure. Let $L^2_a(\mu_h)$ be the weighted Bergman space consisting of holomorphic functions in $L^2(\mu_h)$.

Lemma 5.3. The weighted Bergman space $L^2_a(\mu_h)$ is a complete reproducing kernel Hilbert space.

Proof. First, let us show that evaluation at any point $z \in \mathbb{B}_n$ defines a bounded linear functional on $L^2_a(\mu_h)$. If $h(z) \neq 0$, then by definition, if $f \in L^2_a(\mu_h)$, $fh \in L^2_a(\mathbb{B}_n)$. Therefore

$$|f(z)| = \frac{1}{|h(z)|} |fh(z)| \leq \frac{1}{|h(z)|(1 - |z|^2)^{(n+1)/2}} \|fh\|_{L^2_a(\mathbb{B}_n)}$$

$$= \frac{1}{|h(z)|(1 - |z|^2)^{(n+1)/2}} \|f\|_{L^2_a(\mu_h)}.$$

If $h(z) = 0$, choose a complex line $L$ through $z$ such that $h$ is not identically 0 on $L$. Then $z$ is an isolated zero point of $h$ in $L$. Choose $r > 0$ so that the circle $C_r := \{w \in L : |w - z| = r\}$ does not intersect the zero set of $h$ and is contained in $\mathbb{B}_n$. It is easy to see that evaluations at points in $C_r$ are uniformly bounded. By the Maximum Principal,

$$|f(z)| \leq \max\{|f(w)| : w \in C_r\}.$$

Therefore evaluation at $z$ is also bounded. This proves that $L^2_a(\mu_h)$ is a reproducing kernel Hilbert space.

Now we show that $L^2_a(\mu_h)$ is complete, or equivalently, $L^2_a(\mu_h)$ is closed in $L^2(\mu_h)$. Suppose $(f_n) \subseteq L^2_a(\mu_h)$ and $f_n$ converges to $f \in L^2(\mu_h)$. From the arguments above, it is easy to see that given a compact subset $K$ of $\mathbb{B}_n$, the evaluation functionals at points in $K$ are uniformly bounded. Therefore there exists $C > 0$ such that

$$\sup_{z \in K} |f_n(z) - f_m(z)| \leq C \|f_n - f_m\|_{L^2(\mu_h)}.$$
Hence the sequence of holomorphic functions \( \{f_n(z)\} \) converges uniformly on compact subsets to a holomorphic function \( \bar{f} \) on \( \mathbb{B}_n \). Since \( f_n \to f \), \( f_n \) converges to \( f \) in measure. Therefore \( f = \bar{f} \) almost everywhere. This shows that \( f \in L^2(\mu_{\mathbb{B}_n}) \). \( L^2(\mu_{\mathbb{B}_n}) \) is complete.

**Lemma 5.4.** Suppose \( h \) is as in Theorem 5.1 \( f \in L^2(\mu_{\mathbb{B}_n}) \). For \( 0 < r < 1 \) and \( z \in \mathbb{B}_n \), write \( f_r(z) = f(\rho z) \), \( f_r \) is defined in a neighborhood of \( \mathbb{B}_n \). We have

\[
\int_{\mathbb{B}_n} |h(z)f_r(z)|^2 \, dv_n(z) \leq \int_{\mathbb{B}_n} |h(z)f(z)|^2 \, dv_n(z).
\]

As a consequence, the set of holomorphic functions defined in a neighborhood of \( \mathbb{B}_n \) is dense in \( L^2(\mu_{\mathbb{B}_n}) \).

**Proof.** Apply Theorem 5.1 for \( w = rz \), we get

\[
|h(z)f(rz)| \leq \frac{(1 - r|z|^2)^N}{(1 - r^2|z|^2)^{n+1}} \int_{E(\rho, 1)} |h(\lambda)f(\lambda)| \, dv_n(\lambda)
\]

Therefore

\[
\int_{\mathbb{B}_n} |h(z)f_r(z)|^2 \, dv_n(z) \leq \int_{\mathbb{B}_n} \frac{1}{(1 - r^2|z|^2)^{2n+1}} \left| \int_{E(\rho, 1)} |h(\lambda)f(\lambda)| \, dv_n(\lambda) \right|^2 \, dv_n(z)
\]

\[
\leq \int_{\mathbb{B}_n} \frac{1}{(1 - r^2|z|^2)^{2n+1}} \int_{E(\rho, 1)} |h(\lambda)f(\lambda)|^2 \, dv_n(\lambda) \nu_n(E(\rho, 1)) \, dv_n(z)
\]

By the Fubini’s Theorem, the last integral is equal to

\[
\int_{\mathbb{B}_n} \int_{\{z \in E(\lambda, 1)\}} \frac{1}{(1 - |\eta|^2)^{n+1}} \, dv_n(\lambda) \left| h(\lambda)f(\lambda) \right|^2 \, dv_n(\lambda)
\]

\[
= \int_{\mathbb{B}_n} \frac{1}{(1 - |\eta|^2)^{n+1}} \frac{1}{r^{2n}} \, dv_n(\eta) \left| h(\lambda)f(\lambda) \right|^2 \, dv_n(\lambda)
\]

\[
\leq \int_{\mathbb{B}_n} \left| h(\lambda)f(\lambda) \right|^2 \, dv_n(\lambda).
\]

Here we used the fact that \( \nu_n(E(w, 1)) \approx (1 - |w|^2)^{n+1} \) and that \( 1 - |\eta|^2 \approx 1 - |\lambda|^2 \) whenever \( \eta \in E(\lambda, 1) \) (cf. [23]).

We have proved the inequality. It remains to show that functions defined in a neighborhood of \( \mathbb{B}_n \) is dense. For any \( f \in L^2(\mu_{\mathbb{B}_n}) \), let \( f_n := f_1 - \frac{1}{n+1} \). Then the
sequence of functions \(\{f_n\}\) are defined in a neighborhood of \(\overline{B_n}\). By the previous argument, they are uniformly bounded in \(L^2(\mu_h)\). Therefore there exists a subsequence that converge weakly. Since \(f_n \to f\) pointwisely, the weak limit must be \(f\). Thus \(f\) lies in the weak closure of the subspace of function defined in a neighborhood of \(\overline{B_n}\). By the Hahn-Banach Theorem, \(f\) also belong to the norm closure. This completes the proof. \(\square\)

**Proposition 5.5.** Suppose \(h\) is a holomorphic function defined on a neighborhood of \(\overline{B_n}\), then the principal submodule \([h]\) consists of functions of the form \(hf \in L^2_a(\overline{B_n})\) where \(f\) is a holomorphic function on \(B_n\), i.e.,

\[
[h] = \{fh : fh \in L^2_a(\overline{B_n}), f \in Hol(\overline{B_n})\}.
\]

**Proof.** Define the operator

\[
I : L^2_a(\mu_h) \to L^2_a(\overline{B_n}), \quad f \mapsto fh.
\]

Then \(I\) is an isomorphism. Clearly \(\text{Ran}(I)\) is closed and contains \([h]\). By Lemma 5.4 functions that are holomorphic in a neighborhood of \(\overline{B_n}\) are dense in \(L^2_a(\mu_h)\). Therefore the image of these functions are dense in \(\text{Ran}(I)\). But these images are in \([h]\). This proves that \(\text{Ran}(I) = [h]\). Therefore

\[
[h] = \{fh : fh \in L^2_a(\overline{B_n}), f \in Hol(\overline{B_n})\}.
\]

This completes the proof. \(\square\)

### 5.2 The Geometric Arveson-Douglas Conjecture

There is also a geometric version of the Arveson-Douglas Conjecture in which we consider \(p\)-essential normality of submodules consisting of functions vanishing on a certain zero variety. We begin by a few definitions. See [3] for more details.

**Definition 5.6.** Let \(\Omega\) be a complex manifold. A set \(A \subseteq \Omega\) is called a (complex) analytic subset of \(\Omega\) if for each point \(a \in \Omega\) there is a neighborhood \(U\) of \(a\) and functions \(f_1, \ldots, f_N\) holomorphic in this neighborhood such that

\[
A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.
\]

In particular, \(A\) is said to be principal if there is a holomorphic function \(f\) on \(\Omega\), not identically vanishing on any component of \(\Omega\), such that \(A = Z(f) := \{z \in \Omega : f(z) = 0\}\). The function \(f\) is called a defining function of \(A\) (not to be confused with the defining function of a pseudoconvex domain).

30
For an analytic subset $A$ of $\Omega$, a point $a \in A$ is called a regular point if there is a neighborhood $U$ of $a$ in $\Omega$ such that $A \cap U$ is a complex submanifold of $\Omega$. Otherwise, $a$ is called a singular point. The set of regular points is dense in $A$ and this leads to a definition of dimension at any point.

**Definition 5.7.** Let $A$ be an analytic subset of $\Omega$. The dimension of $A$ at an arbitrary point $a \in A$ is the number

$$\dim_a A := \lim_{z \to a, z \text{ regular}} \dim_z A.$$ 

The dimension of $A$ is, by definition, the maximum of its dimensions at points:

$$\dim A := \max_{z \in A} \dim_z A.$$ 

$A$ is said to be pure if its dimensions at all points coincide.

Pure analytic subsets of codimension 1 have some very important properties.

**Lemma 5.8.** [3 Corollary 1, Page 26] Every pure $(n-1)$-dimensional analytic subset on an $n$-dimensional complex manifold is locally principal, i.e., for any $a \in A$ there exist open neighborhood $U$ of $a$ in $\Omega$ and holomorphic function $f$ on $U$ such that $A \cap U = \{z \in U : f(z) = 0\}$.

Let $A$ be a principal analytic subset of $\Omega$, i.e., $A = \{z \in \Omega : f(z) = 0\}$ for a certain holomorphic function $f$. The function $f$ is called a minimal defining function of $A$ if for every open set $U \subseteq \Omega$ and every $g \in \text{Hol}(U)$ such that $g|_{A \cap U} = 0$, there exists an $h \in \text{Hol}(U)$ such that $g = fh$ in $U$.

**Lemma 5.9.** [3 Proposition 1, Page 27] Every pure $(n-1)$-dimensional analytic subset on an $n$-dimensional complex manifold locally has a minimal defining function.

Now suppose $V$ is a pure $(n-1)$-dimensional analytic subset of an open neighborhood of $\mathbb{B}_n$. Choose $r > 1$ so that $V$ is defined in a neighborhood of $r \mathbb{B}_n$. By Lemma 5.9 and compactness, there is a finite open cover $\{U_i\}$ of $r \mathbb{B}_n$ and a minimal defining function $h_i$ on $U_i$. By definition, if $U_i \cap U_j \neq \emptyset$, the function $g_{ij} = h_i/h_j$ is holomorphic and non-vanishing on $U_i \cap U_j$. They satisfy

$$g_{ij} \cdot g_{ji} = 1 \text{ on } U_i \cap U_j$$

and

$$g_{ij} \cdot g_{jk} \cdot g_{ki} = 1 \text{ on } U_i \cap U_j \cap U_k.$$
Such a set of functions is called a second Cousin data. By [17], the second Cousin problem is solvable on $r\mathbb{B}_n$. That means, there exists non-vanishing $f_i \in Hol(U_i \cap r\mathbb{B}_n)$ such that $g_{ij} = f_i/f_j$. If we define $f = h_i/f_i$ on $U_i \cap r\mathbb{B}_n$, then one easily checks that $f$ is well defined and becomes a global minimal defining function for $V$ in $r\mathbb{B}_n$.

Suppose $f \in L^2_\alpha(\mathbb{B}_n)$, $f|_{V \cap \mathbb{B}_n} = 0$. Then $f = gh$ for some $g \in Hol(\mathbb{B}_n)$. From last subsection we know that this means $[h] \subseteq \{ f \in L^2_\alpha(\mathbb{B}_n) : f|_{V \cap \mathbb{B}_n} = 0 \}$. To sum up, we have obtained the following theorem.

**Theorem 5.10.** Suppose $V$ is a pure $(n - 1)$-dimensional analytic subset of an open neighborhood of $\overline{\mathbb{B}_n}$, then $V$ has a minimal defining function $h$ on an open neighborhood of $\overline{\mathbb{B}_n}$. Moreover, $P_V := \{ f \in L^2_\alpha(\mathbb{B}_n) : f|_{V \cap \mathbb{B}_n} = 0 \} = [h]$. Therefore the submodule $P_V$ is $p$-essentially normal for all $p > n$.

### 5.3 Quotient Modules

Suppose $h$ is a holomorphic function defined on an open neighborhood of $\overline{\mathbb{B}_n}$ and $[h] \subseteq L^2_\alpha(\mathbb{B}_n)$ is the principal submodule generated by $h$. Let $Q_h = [h]^{\perp}$ be the quotient module. We have already showed that $[h]$ is $p$-essentially normal for $p > n$. Classical result [1] shows that $Q_h$ is also $p$-essentially normal for all $p > n$. Let $\mathcal{T}(Q_h)$ be the $C^*$-algebra generated by

$$\{ S_p : p \in \mathbb{C}[z_1, \ldots, z_n] \}$$

where the operator $S_p$ is the compression of $M_p$ to $Q_p$. Note that $\|S_p\| \leq \|M_p\| = \|f\|_\infty$, $\mathcal{T}(Q_h)$ is also generated by

$$\{ S_f : f \text{ is holomorphic in a neighborhood of } \overline{\mathbb{B}_n} \}.$$

By the essential normality of $Q_h$, the quotient $\mathcal{T}(Q_h)/\mathcal{K}(Q_h)$ is a commutative $C^*$-algebra and therefore is isometrically isomorphic to $C(X_h)$ for some compact metrizable space $X_h$. The proof of the following proposition is mostly the same as in [7], the only difference being that instead of considering the polynomial ring, one consider the ring of holomorphic functions defined in a neighborhood of $\overline{\mathbb{B}_n}$.

**Proposition 5.11.** Let $h$ be a holomorphic function defined on an open neighborhood of $\overline{\mathbb{B}_n}$. Let $X_h$ be as above, then

$$\overline{Z(h) \cap \mathbb{B}_n} \cap \partial \mathbb{B}_n \subseteq X_h \subseteq \overline{Z(h) \cap \partial \mathbb{B}_n}.$$

**Acknowledgement** The second author is partially supported by National Natural Science Foundation of China(11371096).
References


Ronald G. Douglas, Department of Mathematics, Texas A&M University, College Station, Texas, 77843, USA, E-mail: rdouglas@math.tamu.edu

Kunyu Guo, School of Mathematical Sciences, Fudan University, Shanghai, 200433, China, E-mail: kyguo@fudan.edu.cn

Yi Wang, Department of Mathematics, Texas A&M University, College Station, Texas, 77843, USA, E-mail: yiwangfdu@gmail.com