Essentially commutative $C^*$–algebras with essential spectrum homeomorphic to $S^{2n-1}$

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Abstract

This paper gives a complete classification of essentially commutative $C^*$–algebras whose essential spectrum is homeomorphic to $S^{2n-1}$ by their characteristic numbers. Let $A_1, A_2$ be such two $C^*$–algebras, then they are $C^*$–isomorphic if and only if they have the same $n$-th characteristic number. Furthermore, let $\gamma_n(A) = m$, then $A$ is $C^*$–isomorphic to $C^*(M_{z_1}, \ldots, M_{z_n})$ if $m = 0$, $A$ is $C^*$–isomorphic to $C^*(T_{z_1}, \ldots, T_{z_{n-1}}, T_{z_m})$ if $m \neq 0$. Some examples are given to show applications of the classification theorem. We finally remark that the proof of the theorem depends on a construction of a complete system of representatives of Ext($S^{2n-1}$).


Key words and phrases: Characteristic numbers, essential spectrum, extensions, mapping degree.

1 Introduction

Let $\mathcal{A}$ be a $C^*$–algebra of operators on a separable Hilbert space $H$. In what follows we assume always that $\mathcal{A}$ contains the identity operator $I$ and the ideal $\mathcal{K}$ of compact operators. We say that $\mathcal{A}$ is essentially commutative if $AB - BA$ is compact for all $A, B \in \mathcal{A}$. A natural problem is how to classify essentially commutative $C^*$–algebras in $C^*$–isomorphism sense. Then the

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problem is to find invariants and models. First if two such $C^*$–algebras
are $C^*$– isomorphic, then the isomorphism is necessarily implemented by a
unitary operator ([Dou]). Let $\mathcal{A}$ be essentially commutative, and $M_A$ be
the maximal ideal space of $\mathcal{A}/\mathcal{K}$ which is called the essential spectrum of $\mathcal{A}$.
For a compact metrizable space $X$, let $\Sigma_X$ denote the class of all essentially
commutative $C^*$–algebras $\mathcal{A}$ whose essential spectrum is homeomorphic to
$X$. Now taking $\mathcal{A}$ in $\Sigma_X$, one hence has a natural extension of $\mathcal{K}$ by $C(X)$

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \overset{\phi}{\rightarrow} C(X) \rightarrow 0.$$ 

The classification problem thus is equivalent to the classification of extensions
of $\mathcal{K}$ by $C(X)$ in the following sense. Let $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$ be two
extensions of $\mathcal{K}$ by $C(X)$. We call them to be weakly equivalent if there
exists the following commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \mathcal{K} \\
\downarrow \theta_1 & & \downarrow \theta_2 \\
\mathcal{A}_1 & \rightarrow & C(X) \\
\downarrow \theta_3 & & \\
0 & \rightarrow & \mathcal{A}_2 \\
\end{array}
$$ 

where $\theta_1$, $\theta_2$ and $\theta_3$ are $C^*$–isomorphisms. Now let $\text{Ext}_w(X)$ denote the
set of the classes of weak equivalence. From Blackadar [Bla], one knows
that $\text{Ext}_w(X)$ is a semigroup, but in general, not a group. Hence intuitively,
the classification problem for $\Sigma_X$ is closely related to the BDF-theory
[BDF1, BDF2] and homotopy theory. For general compact metrizable space
$X$, it is extremely difficult to classify $\Sigma_X$ in $C^*$–isomorphism sense. In [Guo1],
we introduce an invariant called the characteristic number to study essentially
normal operators. In the present paper, we will develop this invariant, and
use it to give $\Sigma_{S^{2n-1}}$ a complete classification, where $S^{2n-1}$ is the boundary
of the unit ball $B_n$ in $\mathbb{C}^n$. For convenience, we write $\Sigma_n$ for $\Sigma_{S^{2n-1}}$. Firstly,
we use the mapping degrees on the unit sphere to give a complete system of
representatives of $\text{Ext}(S^{2n-1})$, and hence shows that the $n$-th characteristic
number $\gamma_n$ is a complete invariant for the class $\Sigma_n$ in $C^*$–isomorphism sense.
Some examples are given to show the applications of the classification theorem.
Since the generalized Poincaré conjecture is true in the case $n \neq 3$ (see
[Sma1, Sam2]), our example shows that Toeplitz algebra $C^*(\Omega)$ on Poincaré
domain $\Omega(\subset \mathbb{C}^n, n \neq 2)$ is necessarily $C^*$–isomorphic to Toeplitz algebra
$C^*(B_n)$ on the unit ball in $\mathbb{C}^n$. In the case $n = 2$, $C^*(\Omega)$ is isomorphic to
$C^*(B_2)$ if and only if the Poincaré conjecture is true for $\partial \Omega$. This fact is
proved by the different method in [Guo2].
2 Some basic lemmas

Let $\mathcal{A}$ be essentially commutative. If a family $\{T_\lambda | \lambda \in \Lambda\}$, $\mathcal{K}$ and the identity operator $I$ generate $\mathcal{A}$, the family $\{T_\lambda | \lambda \in \Lambda\}$ is called a set of generators of $\mathcal{A}$. The rank of $\mathcal{A}$, by definition, is the minimum cardinality of such a family, and is denoted by rank$(\mathcal{A})$. A $C^*$–algebra is said to be finitely generated if rank$(\mathcal{A})$ is finite. Let rank$(\mathcal{A}) = n$, and $\{T_1, T_2, \cdots, T_n\}$ be a set of generators of $\mathcal{A}$. This induces a natural homeomorphism

$$\tau : M_A \to \Delta$$

by $\tau(m) = (\hat{T}_1(m), \cdots, \hat{T}_n(m))$, where $\hat{T}$ denotes the Gelfand transform of $T$ onto $C(M_A)$ and $\Delta = \{(\hat{T}_1(m), \cdots, \hat{T}_n(m)) | m \in M_A \}(\subset C^n)$. It is obvious that the topological dimension of $\Delta(\leq 2n)$ is uniquely determined by $\mathcal{A}$. For the unit sphere $S^{2n-1}$ of $C^n$, we have the following basic fact.

**Lemma 2.1.** Let the essential spectrum $M_A$ of $\mathcal{A}$ be homeomorphic to $S^{2n-1}$, then rank$(\mathcal{A}) = n$, and there exists a set $\{T_1, T_2, \cdots, T_n\}$ of generators of $\mathcal{A}$ such that

$$\tau : M_A \to S^{2n-1}; \quad \tau(m) = (\hat{T}_1(m), \cdots, \hat{T}_n(m))$$

is a homeomorphism.

**Proof.** If the essential spectrum $M_A$ of $\mathcal{A}$ is homeomorphic to $S^{2n-1}$, then one has a natural extension of $\mathcal{K}$ by $C(S^{2n-1})$

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \overset{\phi}{\longrightarrow} C(S^{2n-1}) \longrightarrow 0. \tag{2.1}$$

Now take $T_i$ in $\phi^{-1}(z_i)$ for $i = 1, 2, \ldots, n$. It is easily checked that the family $\{T_1, T_2, \cdots, T_n\}$ is a set of generators of $\mathcal{A}$, and

$$\tau : M_A \to S^{2n-1}; \quad \tau(m) = (\hat{T}_1(m), \cdots, \hat{T}_n(m))$$

is a homeomorphism. Since

$$2 \text{rank}(\mathcal{A}) \geq 2n - 1,$$

this implies rank$(\mathcal{A}) = n$. \qed
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From Lemma 2.1, each $A$ in $\Sigma_n$ yields an extension (2.1) of $K$ by $C(S^{2n-1})$ and hence yields the following exact sequence

$$0 \longrightarrow K \otimes M_k \longrightarrow A \otimes M_k \overset{\phi \otimes 1}{\longrightarrow} C(S^{2n-1}) \otimes M_k \longrightarrow 0$$

for the algebra $M_k$ of $k \times k$ complex matrices. For $A \in A \otimes M_k$, $\tilde{A}$, the image of $A$ in $C(S^{2n-1}) \otimes M_k$, is called the symbol of $A$. It is easily seen that $A$ is Fredholm if and only if $\tilde{A}$ has non-vanishing determinant.

**Lemma 2.2.** Let $n > 1$ and $k < n$. Then for any Fredholm operator $A$ in $A \otimes M_k$, we have $\text{index}(A) = 0$.

**Proof.** Let $GL(n,C)$ denote the complex linear group. Consider a continuous map

$$F : S^{2n-1} \rightarrow GL(n,C)$$

The first column $F_1$ of the matrix $F$ defines a map

$$F_1 : S^{2n-1} \rightarrow C^n \setminus \{0\}$$

so that $f = F_1/|F_1|$ is a map from $S^{2n-1}$ to $S^{2n-1}$. This map has a degree, $\text{deg}(f)$, up to a sign, the number of points in $h^{-1}(p)$, where $h$ is a differentiable approximation to $f$ and $p$ is a general point (see [Ati] or [Hir]). For $F$, we then define the degree of $F$ by

$$\text{deg}(F) = \frac{(-1)^{n-1} \text{deg}(f)}{(n-1)!}$$

Defining $\text{index}(\tilde{A})$ by $\text{index}(A)$, then $\text{index}(\tilde{A}) = \text{index}(\tilde{A}, I_{n-k})$, where $I_{n-k}$ is the $(n-k) \times (n-k)$ identity matrix, and $(\tilde{A}, I_{n-k})$ denotes the matrix

$$(\tilde{A}, I_{n-k}) = \left( \begin{array}{cc} \tilde{A} & 0 \\ 0 & I_{n-k} \end{array} \right).$$

Let $F_1$ be the first column of the matrix $(\tilde{A}, I_{n-k})$. It is obvious that the image of $f = F_1/|F_1| : S^{2n-1} \rightarrow S^{2n-1}$ is a proper closed subset of $S^{2n-1}$. One thus concludes $\text{deg}(f) = 0$ by [BT] or [Hir]. Let continuous maps $\tilde{A}$ and $\tilde{I}_n$ from $S^{2n-1}$ to $GL(n,C)$ be given respectively by $(\tilde{A}, I_{n-k})$ and the $n \times n$ identity matrix $I_n$. Since

$$\text{deg}(\tilde{A}) = \text{deg}(\tilde{I}_n) = 0,$$
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the theorem of Bott implies that $\hat{A}$ can be continuously deformed to $\hat{I}_n$ (see [Ati]). Combining the above discussion with Douglas [Dou], we see that

$$\text{index}(A) = \text{index}(\hat{A}) = \text{index}(\hat{A}, I_{n-k}) = \text{index}(I_n) = \text{index}(I) = 0.$$

□

In [Guo1], we introduced an invariant called the characteristic number to study essentially normal operators. Lemma 2.2 motivates us to introduce characteristic numbers for $C^*$–algebras. For any essentially commutative $C^*$–algebra $A$, since the image of all Fredholm operators in $A$ is a multiplicative group in the Calkin algebra, it follows that the indices of all Fredholm operators in $A$ form a subgroup $\Gamma$ of the integer group $\mathbb{Z}$, that is, there exists a unique non-negative integer $m$ such that $\Gamma = m\mathbb{Z}$. The characteristic number $\gamma(A)$ of $A$, by definition, is the above $m$. We also define the $n$-th characteristic number $\gamma_n(A)$ of $A$ by $\gamma(A \otimes M_n)$. By the inclusion of $A \otimes M_n$ in $A \otimes M_{n+1}$ which sends $A$ to $(A, I)$, this forces that $\gamma_{n+1}(A)$ is a factor of $\gamma_n(A)$ for any natural number $n$.

**Lemma 2.3.** Let $A$ be in the class $\Sigma_n$. Then

$$\gamma_1(A) = \gamma_2(A) = \cdots = \gamma_{n-1}(A) = 0$$

and

$$\gamma_n(A) = \gamma_{n+1}(A) = \cdots.$$

**Proof.** From Lemma 2.2, we only need to show that

$$\gamma_n(A) = \gamma_{n+1}(A) = \cdots.$$

Let $F : S^{2n-1} \rightarrow GL(N, C)$ be a continuous map, here $N > n$. Then by Atiyah [Ati], there is a continuous map

$$G : [0, 1] \times S^{2n-1} \rightarrow GL(N, C)$$

such that $G(0, z) = F(z)$ and

$$G(1, z) = \begin{pmatrix} H(z) & 0 \\ 0 & I_{N-n} \end{pmatrix},$$

where $I_{N-n}$ is the $(N-n) \times (N-n)$ identity matrix. The argument used in the proof of Lemma 2.2 can then be exploited to show that $\gamma_N(A) = \gamma_n(A)$.

□

To understand the importance of characteristic numbers a little better we shall see in section 3 that $\gamma_n$ is a complete invariant of $C^*$–algebras in $\Sigma_n$ in $C^*$–isomorphism sense.
3 The equivalence classes of $\mathrm{Ext}(S^{2n-1})$

Let us begin with facts from the BDF-theory [BDF1, BDF2]. Let $X$ be a compact metrizable space. An extension of $\mathcal{K}$ by $C(X)$ is a pair $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a $C^*$-subalgebra of operators on some separable Hilbert space which contains $\mathcal{K}$ and the identity operator $I$, and $\phi$ is a $C^*$-homomorphism of $\mathcal{E}$ onto $C(X)$ with kernel $\mathcal{K}$. In the language of homology, an extension $(\mathcal{E}, \phi)$ is a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \xrightarrow{\phi} C(X) \rightarrow 0.$$ 

Extensions $(\mathcal{E}_1, \phi_1)$ and $(\mathcal{E}_2, \phi_2)$ are called equivalent if there exists a $C^*$-isomorphism $\psi : \mathcal{E}_2 \rightarrow \mathcal{E}_1$ such that $\phi_2 = \phi_1 \psi$. The set of equivalence classes of extensions of $\mathcal{K}$ by $C(X)$ is denoted $\mathrm{Ext}(X)$. In [BDF2], they proved that $\mathrm{Ext}(X)$ is a group, and the correspondence $X \mapsto \mathrm{Ext}(X)$ yields a homotopy invariant covariant functor. It is well known that one of the applications of the BDF-theory is to classify essentially normal operators modulo the compacts under unitary equivalence (see [BDF1]).

Below, we shall concentrate on working out explicitly a complete system of representatives for the equivalence classes of extensions of $\mathcal{K}$ by $C(S^{2n-1})$. In the case $n = 1$, a complete system of representatives of $\mathrm{Ext}(S^1)$ is worked out explicitly by Toeplitz extension on the unit circle $S^1$ in [BDF1]. In the case $n > 1$, the periodicity theorem of $\mathrm{Ext}$ ([BDF2]) implies $\mathrm{Ext}(S^{2n-1}) = \mathbb{Z}$. Let $L^2(\mathbb{B}_n)$ be the Bergman space on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, and let $C^*(\mathbb{B}_n)$ be the $C^*$-algebra generated by all Toeplitz operators on $L^2(\mathbb{B}_n)$ with symbols in $C(\overline{\mathbb{B}_n})$. The Coburn exact sequence ([Cob])

$$0 \rightarrow \mathcal{K} \rightarrow C^*(\mathbb{B}_n) \xrightarrow{\pi} C(S^{2n-1}) \rightarrow 0$$

is a natural extension of $\mathcal{K}$ by $C(S^{2n-1})$. From Venugopalkrishna [Ven], we see that this extension is a generator of $\mathrm{Ext}(S^{2n-1})$ (see also [BDF2]). By the theorem of Bott in [Ati], there is a natural isomorphism

$$\pi_{2n-1}(GL(n, \mathbb{C})) \cong K^1(S^{2n-1}) \cong \mathbb{Z},$$

where $\pi_{2n-1}(GL(n, \mathbb{C}))$ is the group of homotopy classes of continuous maps from $S^{2n-1}$ to $GL(n, \mathbb{C})$. Applying the BDF-theory [BDF2], the homomorphism

$$\gamma_\infty : \mathrm{Ext}(S^{2n-1}) \rightarrow \mathrm{Hom}(\pi_{2n-1}(GL(n, \mathbb{C})), \mathbb{Z}) \cong \mathbb{Z}$$
is surjective, and hence is an isomorphism. For an extension \((E, \phi), \gamma_\infty(E)\) is defined by
\[
\gamma_\infty(E)[f_{ij}] = \text{index } [\phi^{-1}(f_{ij})].
\]

Let \(L^2(S^{2n-1})\) denote the Hilbert space of square-integrable functions on \(S^{2n-1}\). For \(f \in C(S^{2n-1})\), we denote by \(M_f\) multiplication operator on \(L^2(S^{2n-1})\). It is well known that \((E_0, \pi_0)\) is the zero element in \(\text{Ext}(S^{2n-1})\), where
\[
E_0 = C^*(M_{z_1}, \ldots, M_{z_n}) = \{M_f + K \mid f \in C(S^{2n-1}), K \in \mathcal{K}\},
\]
and \(\pi_0(M_f + K) = f\). Now let \(\sigma : S^{2n-1} \to S^{2n-1}\) be a continuous map with the mapping degree \(\text{deg}(\sigma) \neq 0\), \(\sigma\) is then surjective. We use \(E_\sigma\) to denote the \(C^*\)-algebra \(\{T_{f_o\sigma} + K \mid f \in C(S^{2n-1}), K \in \mathcal{K}\}\), where \(T_{f_o\sigma}\) is Toeplitz operator with symbol \(f \circ \sigma\) on the Bergman space \(L^2_n(B_n)\), and \(f \circ \sigma\) is the standard Poisson extension of \(f \circ \sigma\) onto \(\overline{B_n}\). This gives an extension \((E_\sigma, \pi_\sigma)\), where \(\pi_\sigma(T_{f_o\sigma} + K) = f\). In fact, it is easily seen that \((E_\sigma, \pi_\sigma) = (C^*(B_n), \pi)\), and hence if \(\sigma_1\) and \(\sigma_2\) are homotopic, the homotopy invariance of \(\text{Ext}\) then implies that \((E_{\sigma_1}, \pi_{\sigma_1})\) and \((E_{\sigma_2}, \pi_{\sigma_2})\) are equivalent.

For \(i = \pm 1, \pm 2, \ldots\), take \(\sigma_i\) to be a continuous map from \(S^{2n-1}\) to \(S^{2n-1}\) with mapping degree \(\text{deg}(\sigma_i) = i\). We have thus the following.

**Lemma 3.1.** The extensions \((E_{\sigma_i}, \pi_{\sigma_i}) (i = \pm 1, \pm 2, \ldots)\) together with the trivial extension \((E_0, \pi_0)\) form a complete system of representatives of \(\text{Ext}(S^{2n-1})\).

**Proof.** Let \(m\) be a non-zero integer. By [BDF2], one has
\[
\gamma_\infty(E_{\sigma_m})[f_{ij}] = \text{index } [T_{ij \circ \sigma_m}] = \text{deg}(\sigma_m) \text{ index } [T_{ij}] = m \text{ index } [T_{ij}],
\]
and
\[
\gamma_\infty(m C^*(B_n))[f_{ij}] = m \text{ index } [T_{ij}].
\]
It follows that \((E_{\sigma_m}, \pi_{\sigma_m})\) and \((m C^*(B_n), \pi^{(m)})\) are equivalent. Note that the Toeplitz extension \((C^*(B_n), \pi)\) is a generator of \(\text{Ext}(S^{2n-1})\), we thus conclude that the extensions \((E_{\sigma_i}, \pi_{\sigma_i})(i = \pm 1, \pm 2, \ldots)\) together with the extension \((E_0, \pi_0)\) form a complete system of representatives of \(\text{Ext}(S^{2n-1})\).

**Lemma 3.2.** Let \(\sigma : S^{2n-1} \to S^{2n-1}\) be a continuous map. Then we have
\[
\gamma_n(E_\sigma) = |\text{deg}(\sigma)|.
\]
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Proof. Apply Venugopalkrishna [Ven, Theorem 1.5] and the multiplication formula of mapping degree [Hir].

Lemma 3.3. Let $\sigma'$, $\sigma'': S^{2n-1} \to S^{2n-1}$ be continuous maps. Then $\mathcal{E}_{\sigma'}$ and $\mathcal{E}_{\sigma''}$ are $C^*$–isomorphic if and only if

$$|\deg(\sigma')| = |\deg(\sigma'')|.$$ 

Proof. If $\mathcal{E}_{\sigma'}$ and $\mathcal{E}_{\sigma''}$ are $C^*$–isomorphic, then the isomorphism is implemented by a unitary operator. This implies thus that

$$\gamma_n(\mathcal{E}_{\sigma'}) = \gamma_n(\mathcal{E}_{\sigma''})$$

and hence by Lemma 3.2,

$$|\deg(\sigma_1)| = |\deg(\sigma_2)|.$$ 

If $\deg(\sigma') = \deg(\sigma'')$, then Hopf lemma ([Hir]) implies that the maps $\sigma'$, $\sigma'': S^{2n-1} \to S^{2n-1}$ are homotopic, and hence the homotopy invariance of Ext shows that $\mathcal{E}_{\sigma'}$ and $\mathcal{E}_{\sigma''}$ are isomorphic as $C^*$–algebras. If $\deg(\sigma') = -\deg(\sigma'')$, write $\sigma'' = (\phi_1, \phi_2 \cdots, \phi_n)$ and define $\sigma'''$ by $$(\phi_1, \phi_2 \cdots, \phi_n),$$

then

$$\deg(\sigma') = \deg(\sigma''')$$

and hence $\mathcal{E}_{\sigma'}$ and $\mathcal{E}_{\sigma'''}$ are $C^*$–isomorphic. Since $\mathcal{E}_{\sigma''} = \mathcal{E}_{\sigma'''}$, it follows that $\mathcal{E}_{\sigma'}$ and $\mathcal{E}_{\sigma''}$ are $C^*$–isomorphic.

Theorem 3.4. Let $\mathcal{E}$ and $\mathcal{F}$ be in $\Sigma_n$. Then $\mathcal{E}$ and $\mathcal{F}$ are $C^*$–isomorphic if and only if

$$\gamma_n(\mathcal{E}) = \gamma_n(\mathcal{F}).$$

Proof. Assume first that $\mathcal{E}$ and $\mathcal{F}$ are $C^*$–isomorphic. Then the equality $\gamma_n(\mathcal{E}) = \gamma_n(\mathcal{F})$ is immediate. Conversely, since $\mathcal{E}$ and $\mathcal{F}$ are respectively $C^*$–isomorphic to one of $\mathcal{E}_0$, $\mathcal{E}_{\sigma_1}, \mathcal{E}_{\sigma_2} \cdots$, Lemmas 3.1, 3.2 and 3.3 imply that if $\gamma_n(\mathcal{E}) = \gamma_n(\mathcal{F})$, then $\mathcal{E}$ and $\mathcal{F}$ are isomorphic as $C^*$–algebras.

From Theorem 3.4, we see that $n$-th characteristic number is a complete invariant for $C^*$–algebras in $\Sigma_n$ in $C^*$–isomorphism sense. In next section we will give examples to show applications of the classification Theorem 3.4.

Now we consider Toeplitz algebras on pseudoregular domains ($\subset \mathbb{C}^n$) with smooth boundary. As pointed out in [Sal, SSU], pseudoregular domains
include the strongly pseudoconvex domains, pseudoconvex domains with real analytic boundary, and more generally, domains of finite type. Let \( \Omega \) be pseudoregular domain with smooth boundary. Following [Sal, SSU], on the Bergman space \( L^2_a(\Omega) \), the \( C^* \)–algebra \( C^*(\Omega) \) generated by Toeplitz operators with symbols in \( C(\Omega) \) is essentially commutative, and its essential spectrum is \( \partial \Omega \). In [SSU], they proved that for each \( \lambda \in \Omega \), Toeplitz tuple \( T_z - \lambda = \{ T_{z_1} - \lambda_1, \ldots, T_{z_n} - \lambda_n \} \) is Fredholm, and \( \text{index}(T_z - \lambda) = (-1)^n \). By [Cur] and Lemma 2.3, one sees that if \( \partial \Omega \) is homeomorphic to \( S^{2n-1} \), then \( \gamma_n(C^*(\Omega)) = 1 \). Since \( \gamma_n(C^*(B_n)) = 1 \), Theorem 3.4 immediately yields the following.

**Example 1.** Let \( \Omega \) be a pseudoregular domain with smooth boundary. Then \( \partial \Omega \) and \( S^{2n-1} \) are homeomorphic if and only if \( C^*(\Omega) \) is isomorphic to \( C^*(B_n) \) as \( C^* \)–algebras.

For a pseudoregular domain \( \Omega \) in \( \mathbb{C}^n \), we say that \( \Omega \) is a Poincaré domain if its boundary \( \partial \Omega \) is homotopy equivalent to the unit sphere \( S^{2n-1} \), that is, \( \partial \Omega \) is a homotopy \( (2n-1) \)–sphere. The generalized Poincaré conjecture says if every closed \( n \)–manifold \( M \) which is a homotopy \( n \)–sphere is homeomorphic to the \( n \)–sphere (see [Sma1, Sam2]). Smale [Sam2] showed that the generalized Poincaré conjecture is true in the case \( n > 4 \). Freedman [Fre] proved the case \( n = 4 \). For \( n = 1, 2 \), it is well known that the generalized conjecture is true (see [Hir]). Therefore the famous Poincaré conjecture says that every closed \( 3 \)–manifold which is a homotopy \( 3 \)–sphere is homeomorphic to the \( 3 \)–sphere. This has never been answered. Therefore, for each Poincaré domain \( \Omega \) in \( \mathbb{C}^n(n \neq 2) \), its boundary \( \partial \Omega \) is actually homeomorphic to the \( (2n-1) \)–sphere. Example 1 shows thus the following.

**Example 2.** Let \( \Omega \) be a Poincaré domain in \( \mathbb{C}^n(n \neq 2) \). Then

\[
C^*(\Omega) \cong C^*(B_n)
\]

Example 2 is proved by different method in [Guo2]. Example 2 suggests that for the Poincaré conjecture in the case of \( \partial \Omega \), an operator algebraic proof is perhaps possible. Of course, the validity of the Poincaré conjecture for \( \partial \Omega \) remains unknown.

**Example 3.** Let \( 0 < p, q < \infty \) and \( \Omega_{p,q} = \{ z \in \mathbb{C}^2 | |z_1|^p + |z_2|^q < 1 \} \). \( \Omega_{p,q} \) is pseudoconvex (because \( \log(\Omega_{p,q})_+ \) is convex); when \( p, q \geq 2 \), \( \Omega_{p,q} \) is Levi pseudoconvex; and \( \Omega_{p,q} \) is strongly pseudoconvex if and only if \( p = q = 2 \). From [CS], on the Bergman space \( L^2_a(\Omega_{p,q}) \), one sees that Toeplitz
algebra $C^*(\Omega_{p,q})$ (generated by Toeplitz operators with symbols in $C(\overline{\Omega}_{p,q})$) is essentially commutative, and its essential spectrum is $\partial\Omega_{p,q}$, also for each $\lambda \in \Omega_{p,q}$, $\text{index}(T_z - \lambda) = 1$. Then by the radial projection, $\partial\Omega_{p,q}$ and $S^3$ are homeomorphic, and hence $C^*(\Omega_{p,q})$ and $C^*(B_2)$ are isomorphic as $C^*$–algebras. However, for $p$ or $q \neq 2$, it is easy to prove that there does not exist any proper holomorphic mapping that maps the unit ball $B_2$ onto $\Omega_{p,q}$. In this example, we restricted ourselves to $n = 2$, but it is clear that all results hold for $n \geq 2$.

**Example 4.** Considering the domain $\Omega = \{ z \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 + |z_1z_2|^2 < 1 \}$, it is easy to check that $\Omega$ is a strongly pseudoconvex domain with smooth boundary. Then by the radial projection, $\partial\Omega$ and $S^3$ are homeomorphic, and hence $C^*(\Omega)$ and $C^*(B_2)$ are isomorphic as $C^*$–algebras. However, by the Cartan Theorem, the unit ball $B_2$ and $\Omega$ are never holomorphically equivalent.

### 4 The construction of representatives of the class $\Sigma_n$

In this section, we shall construct explicitly a complete system of representatives of the class $\Sigma_n$. Let $m$ be a positive integer. Define the map $\sigma_m : S^{2n-1} \rightarrow S^{2n-1}$ by

$$\sigma_m(z_1, \cdots, z_n) = \left( z_1, \cdots, z_{n-1}, \frac{|z_n|^m}{|z_n|^{m}} \right)$$

and the map $\sigma_{-m} : S^{2n-1} \rightarrow S^{2n-1}$ by

$$\sigma_{-m}(z_1, \cdots, z_n) = \left( z_1, \cdots, z_{n-1}, \frac{|z_n|^m}{|z_n|^{m}} \right).$$

We claim $\deg(\sigma_m) = m$; $\deg(\sigma_{-m}) = -m$.

Write $\Omega_m$ for pseudoconvex domain

$$\{(z_1, \cdots, z_n) | |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^\frac{2}{m} < 1 \}.$$ 

The domain above is pseudoconvex because $\log(\Omega_m)_+$ is convex. Denote by $\partial\Omega_m$ the boundary of $\Omega_m$, that is,

$$\partial\Omega_m = \{(z_1, \cdots, z_n) | |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^\frac{2}{m} = 1 \}.$$
Since $\overline{\Omega}_m$ has the standard orientation, its boundary $\partial \Omega_m$ inherits an orientation (except some special points), also called “standard”. This means that $(e_1, \cdots, e_{2n-1})$ is an orienting basis for $\partial \Omega_m$ if $(e_1, \cdots, e_{2n-1}, e_{2n})$ is an orienting basis for $\overline{\Omega}_m$ and $e_{2n}$ points into $\Omega_m$ at $z \in \partial \Omega_m$. Define a map $\delta_m$ from $S^{2n-1}$ onto $\partial \Omega_m$ by

$$\delta_m(z_1, \cdots, z_n) = (z_1, \cdots, z_{n-1}, z_m^m).$$

A straightforward calculation of Jacobi matrix yields that the mapping degree $\deg(\delta_m) = m$. Furthermore, we establish an orientation-preserving homeomorphism $\eta_m : \partial \Omega_m \to S^{2n-1}$ by

$$\eta_m(z_1, \cdots, z_{n-1}, z_n) = (z_1, \cdots, z_{n-1}, |z_n|^\frac{1}{m} \bar{z}_m).$$

It is easily checked that

$$\sigma_m = \eta_m \circ \delta_m$$

and hence

$$\deg(\sigma_m) = \deg(\eta_m) \deg(\delta_m) = 1 \cdot m = m.$$  

Similarly, define an anti-orientation homeomorphism $\eta_m : \partial \Omega_m \to S^{2n-1}$ by

$$\eta_{-m}(z_1, \cdots, z_{n-1}, z_n) = (z_1, \cdots, z_{n-1}, |z_n|^\frac{1}{m} \bar{z}_n).$$

It is easily seen that $\sigma_{-m} = \eta_{-m} \circ \delta_m$, and $\deg(\sigma_{-m}) = -m$ follows. The claim is proved.

Let $C^*(T_{z_1}, \cdots, T_{z_{n-1}}, T_{z_m})$ be the $C^*$-algebra generated by $T_{z_1}, \cdots, T_{z_{n-1}}, T_{z_m}$, the identity operator and all compact operators on the Bergman space $L^2_a(B_n)$. We define extensions

$$(C^*(T_{z_1}, \cdots, T_{z_{n-1}}, T_{z_m}), \pi_m)$$

and

$$(C^*(T_{z_1}, \cdots, T_{z_{n-1}}, T_{z_m}), \pi_{-m})$$

of $\mathcal{K}$ by $C(S^{2n-1})$ respectively by

$$\pi_m(T_f(z_1, \cdots, z_{n-1}, z_m) + K) = f|_{\partial \Omega_m} \circ \eta_m^{-1}$$

and

$$\pi_{-m}(T_f(z_1, \cdots, z_{n-1}, z_m) + K) = f|_{\partial \Omega_m} \circ \eta_{-m}^{-1}$$

here $f \in C(\overline{\Omega}_m)$, $K \in \mathcal{K}$.

We are now in a position to give a main result in this section.

**Theorem 4.1.**
(1) \( \mathcal{E}_{\sigma_m} = \mathcal{E}_{\sigma_{-m}} = C^*(T_{z_1}, \cdots , T_{z_{n-1}}, T_{z_m}) \).

(2) The extensions

\[
(C^*(T_{z_1}, \cdots , T_{z_{n-1}}, T_{z_m}), \pi_m) \quad \text{and} \quad (C^*(T_{z_1}, \cdots , T_{z_{n-1}}, T_{z_m}), \pi_{-m})
\]

(here \( m = 1, 2, \ldots \)) and the trivial extension \( (\mathcal{E}_0, \pi_0) \) form a complete system of representatives of \( \text{Ext}(S^{2n-1}) \).

Proof. (1). It is obvious that the relation \( \mathcal{E}_{\sigma_m} = \mathcal{E}_{\sigma_{-m}} \) is true. Then an operator \( A \in \mathcal{E}_{\sigma_m} \), if and only if \( A \) has form

\[
A = T_{\tilde{f}} \circ \sigma_m + \text{compact}, \quad f \in C(S^{2n-1}),
\]

and \( B \in C^*(T_{z_1}, \cdots , T_{z_{n-1}}, T_{z_m}) \) if and only if \( B \) has form

\[
B = T_{f(z_1, \cdots , z_{n-1}, z_m)} + \text{compact}, \quad f \in C(\Omega_m).
\]

The orientation-preserving homeomorphism \( \eta_m \) and the relation \( \sigma_m = \eta_m \circ \delta_m \) imply then

\[
\mathcal{E}_{\sigma_m} = C^*(T_{z_1}, \cdots , T_{z_{n-1}}, T_{z_m}).
\]

(2). Apply Lemma 3.1 and the above (1). \( \square \)

From Lemma 3.2, Theorem 3.4, and Theorem 4.1, we immediately obtain the following:

**Corollary 4.2.** Let \( \mathcal{E} \in \Sigma_n \). Then

(1). if \( \gamma_n(\mathcal{E}) = 0 \), then \( \mathcal{E} \) and \( \mathcal{E}_0 \) are \( C^* \)-isomorphic;

(2). if \( \gamma_n(\mathcal{E}) = m > 0 \), then \( \mathcal{E} \) and \( C^*(T_{z_1}, \cdots , T_{z_{n-1}}, T_{z_m}) \) are \( C^* \)-isomorphic.

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**References**


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