THE CODIMENSION FORMULA ON AF–COSUBMODULES

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ABSTRACT. Let $M_1$, $M_2$ be submodules of analytic Hilbert module $X$ on $\Omega(\subset C^n)$ such that $M_1 \supseteq M_2$ and $\dim M_1/M_2 = k < \infty$. If $M_2$ is an AF-cosubmodule, then the codimension $\dim M_1/M_2$ of $M_2$ in $M_1$ equals the cardinality of zeros of $M_2$ related to $M_1$ by counting multiplicities. The codimension formula has some interesting applications.

§1. Introduction

The Beurling’s theorem shows that all submodules of the Hardy module $H^2(D)$ on the unit disk $D$ are isomorphic [9]. In generalizing this result to several variables, one was somewhat surprised to find that the analogous result is false[3,4,8,9]. Due to the extreme complexity of the structure of analytic submodules of several variables, the additional assumption on finite codimension was naturally adopted as a first step towards a better understanding of their properties [4,5,6,7].

In [1], we study the structure of zero varieties of Hardy-submodules generated by polynomials. In [2], we developed the characteristic space theory of analytic Hilbert modules to study algebraic reduction and rigidity of Hilbert modules. In this note, using the characteristic space theory, we generalize the main results in [1] to the case of AF-cosubmodules of analytic Hilbert modules of several variables.

§2. Preliminary notations

Let us recall some basic notations in [2]. Let $\Omega$ be a bounded nonempty open subset of $C^n$, $Hol(\Omega)$ denote the ring of analytic functions on $\Omega$, and $X$ be Banach space contained in $Hol(\Omega)$. We call $X$ to be a reproducing $\Omega$-space if $X$ contains 1 and if for each $w \in \Omega$ the evaluation function, $E_w(f) = f(w)$, is a continuously linear functional on $X$. Write $C$ for the ring of all polynomials on $C^n$. We call $X$ a reproducing $C$-module on $\Omega$ if $X$ is a reproducing $\Omega$–space, and for each polynomial $p$ and each $x \in X$, $p\, x$ is contained in $X$. Note that, by a simple application of the

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closed graph theorem, the operator $T_p$ defined to be multiplication by $p$ is bounded on $X$ for each $p \in \mathcal{C}$. Note also that $\mathcal{C} \subset X$ follows from the fact that 1 is in $X$. For $w \in \mathbb{C}^n$, we call $w$ to be a virtual point of $X$ provided that the linear functional $f \mapsto f(w)$ defined on $\mathcal{C}$ extends to a bounded linear function on $X$. We use $vp(X)$ to denote the collection of virtual points, then $vp(X) \supseteq \Omega$. We say that $X$ is an analytic Hilbert module on $\Omega$ if the following conditions are satisfied:

1. $X$ is a reproducing $\mathcal{C}$-module on $\Omega$;
2. $\mathcal{C}$ is dense in $X$;
3. $vp(X) = \Omega$.

For a polynomial $q = \sum a_{m_1\cdots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$, let $q(D)$ denote the linear partial differential operator $\sum a_{m_1\cdots m_n} \partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}$. Let $M$ be a submodule of $X$, and $\lambda \in \Omega$. Set

$$M_\lambda = \{ q \in \mathcal{C} | q(D)f|\lambda = 0, \forall f \in M \}$$

where $q(D)f|\lambda$ denotes $(q(D)f)(\lambda)$. From Leibnitz rule, for any polynomial $q$ and any analytic function $f$, we have the following:

$$q(D)(z_j f)|\lambda = \lambda_j q(D)f|\lambda + \frac{\partial q}{\partial z_j}(D)f|\lambda \quad j = 1, 2, \cdots, n$$

By the above equality, $M_\lambda$ is thus invariant under the action by basic partial differential operators $\{ \partial / \partial z_1, \partial / \partial z_2 \cdots, \partial / \partial z_n \}$, and $M_\lambda$ is called to be the characteristic space of $M$ at $\lambda$. The envelope of $M$ at $\lambda$, $M_\lambda^e$, is defined by

$$M_\lambda^e = \{ f \in X | q(D)f|\lambda = 0, \forall q \in M_\lambda \}$$

According to Leibnitz rule, it is not difficult to see that $M_\lambda^e$ is a submodule of $X$, and $M_\lambda^e \supseteq M$.

Let $M$ be a submodule of $X$. We call $M$ to be approximately finite codimensional (abbr., AF-cosubmodule) if $M$ is equal to the intersection of all finite codimensional submodules containing $M$. The reason is that in this case, $M$ is just the limit of decreasing net ($\supseteq$) of all finite codimensional submodules containing $M$. For a submodule $M$, the AF-envelope of $M$ is defined by the intersection of all finite codimensional submodules containing $M$, and denoted by $M^e$. Clearly, the definition implies that the envelope of a submodule $M$ is an AF-cosubmodule. In what follows we will use $Z(M)$ to denote the zero set of $M$, that is, $Z(M) = \{ \lambda \in \mathbb{C}^n | f(\lambda) = 0, \forall f \in M \}$. The next lemma illustrates some basic properties of characteristic space and envelope (see[2]).
Lemma 2.1[2]. Let $X$ be an analytic Hilbert module, and $M$ a submodule of $X$. Then we have

1. if $Z(M) = \emptyset$, then $M^e = X$;
2. if $Z(M) \neq \emptyset$, then $M \subseteq M^e \neq X$, $(M^e)^e = M^e$, and $Z(M) = Z(M^e)$;
3. if $Z(M) \neq \emptyset$, then $M^e = \bigcap_{\lambda \in Z(M)} M^e_{\lambda}$.

In particular, let $M_1, M_2$ be two submodules of $X$, then $M_1^e = M_2^e$ if and only if $Z(M_1) = Z(M_2)$, and for every $\lambda \in Z(M_1)$, $M_{1\lambda} = M_{2\lambda}$.

§3. The codimension formula on AF–cosubmodules

Let $M_1, M_2$ be submodules of $X$, and $\lambda \in \Omega$. We call that $M_1, M_2$ have the same multiplicity at $\lambda$ if $M_{1\lambda} = M_{2\lambda}$. Let the symbol $Z(M_2) \setminus Z(M_1)$ denote the set of zeros of $M_2$ related to $M_1$, that is, $Z(M_2) \setminus Z(M_1)$ is defined by $\{\lambda \in Z(M_2) \mid M_{2\lambda} \neq M_{1\lambda}\}$. If $M_1 \supseteq M_2$, the cardinality of zeros of $M_2$ related to $M_1$, $\text{card}(Z(M_2) \setminus Z(M_1))$, is defined by $\sum_{\lambda \in Z(M_2) \setminus Z(M_1)} \dim M_{2\lambda}/M_{1\lambda}$. Combining the technique used in [1] with Lemma 2.1, we have

Theorem 3.1. Let $M_1, M_2$ be submodules of analytic Hilbert module $X$ on $\Omega$ such that $M_1 \supseteq M_2$ and $\dim M_1/M_2 = k < \infty$. If $M_2$ is an AF–cosubmodule. Then we have

1. $Z(M_2) \setminus Z(M_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subseteq \Omega$,
2. $M_2 = \{h \in M_1 \mid p(D)h|_{\lambda} = 0, p \in M_{2\lambda}, \lambda \in Z(M_2) \setminus Z(M_1)\}$,
3. $\dim M_1/M_2 = \text{card}(Z(M_2) \setminus Z(M_1))$

where $(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$ is $n$-tuple of operators which are defined on the quotient module $M_1/M_2$ by $M_{z_i}f = (z_i f)$ for $i = 1, \cdots, n$, and $\sigma_p(M_{z_1}, \cdots, M_{z_n})$ denotes the joint eigenvalues of the $n$–tuple $(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$. It is worth noticing that (3) of Theorem 3.1 says the codimension $\dim M_1/M_2$ of $M_2$ in $M_1$ equals the cardinality of zeros of $M_2$ related to $M_1$. In this way, (3) is the codimension formula what we say.

Remark. It is worth noticing that the assumption is necessary in Theorem 3.1 that $M_2$ is approximately finite codimensional. In fact, by [10], we know that there exists a submodule $M$ of the Bergman module $L^2_2(D)$ of the unit disk $D$ such that $\dim M/zM = 2$, while $Z(zM) \setminus Z(M) = \{0\}$ and $\dim [(zM)0/M_0] = 1$.

Now assume that $M$ is a finite codimensional submodule of $X$. Then Theorem 3.1 implies the following:

Corollary 3.2. Let $M$ be a finite codimensional submodule of $X$, then we have

1. $Z(M) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subseteq \Omega$,
2. $M = \bigcap_{\lambda \in Z(M)} M^e_{\lambda}$.
\[ (3) \quad \text{codim} M = \text{card}(Z(M)) = \sum_{\lambda \in Z(M)} \text{dim} M_{\lambda}. \]

Notice that (3) of Corollary 3.2 says that the codimension \( \text{codim} M \) of \( M \) in \( X \) equals the cardinality of zeros of \( M \) by counting multiplicities.

From Theorem 3.1 and the technique used in [1], we also have the following

**Theorem 3.3.** Let \( M_1, M_2 \) be submodules of analytic Hilbert module \( X \) on \( \Omega \) such that \( M_1 \supseteq M_2 \) and \( \text{dim} M_1/M_2 = k < \infty \). If \( M_1 \) is an AF–cosubmodule. Then we have

\[ \text{dim} M_1/M_2 \geq \text{card}(Z(M_2) \setminus Z(M_1)), \]

and “equal” if and only if \( M_2 \) is an AF–cosubmodule.

In what follows we give three examples to show applications of the results in this note.

**Example 1.** Let \( I \) be a finite codimensional ideal of polynomials on \( C^m \), then \( I \) has an irredundant primary decomposition in \( C \) (see [11]): \( I = \bigcap_{j=1}^{m} I_j \), where \( I_j \) is primary for a maximal ideal of evaluation at some point \( \lambda_j \). Since \( I_i + I_j = C \) for \( i \neq j \), it follows that \( I = \prod_{j=1}^{m} I_j \). This derives that for any natural number \( k \),

\[ I^k = \prod_{j=1}^{m} I_j^k = \bigcap_{j=1}^{m} I_j^k. \]

So, \( \text{codim} I^k = \sum_{j=1}^{m} \text{codim} I_j^k \). As is well known that for large integer \( k \), \( \text{codim}(I_j^k) \) is a polynomial of \( k \) with the degree \( n \) which is called the Hilbert–samuel polynomial of \( I_j \) (see [11]). Therefore, for each finite codimensional ideal \( I \), the codimension of \( I^k \), \( \text{codim}(I^k) \), is a polynomial of \( k \) with the degree \( n \) which is said to be Hilbert–samuel polynomial of \( I \), and denoted by \( P_I(k) \). Let \( X \) be an analytic Hilbert module on \( \Omega \), and \( I \) be an ideal of \( C \). It is easily known that \( I \) can be uniquely decomposed into \( I_{\Omega} \cap I_{\Omega^c} \) such that the zeros of \( I_{\Omega} \) are in \( \Omega \), while those of \( I_{\Omega^c} \) do not. Write \( [I] \) for the closure of \( I \) in \( X \), then \( [I] \) is a submodule of \( X \). Let \( I \) be a finite codimensional ideal. Then for any natural number \( k \), one has \( [I^k] = [I_{\Omega}^k]. \) From Corollary 2.8 in [8], the equality \( \text{codim} [I^k] = \text{codim} I_{\Omega}^k \) is immediate. From Corollary 3.2, we have the following:

Let \( I \) be a finite codimensional ideal. Then for large integer \( k \), the cardinality of zeros of \( [I^k] \), \( \text{card}(Z([I^k])) \), is a polynomial of \( k \) with the degree \( n \), more precisely,

\[ \text{card}(Z([I^k])) = P_{I_{\Omega}}(k). \]

**Example 2.** Recall that Rudin’s submodule \( M \) of \( H^2(D^2) \) over the bidisk is defined to be the collection of all functions in \( H^2(D^2) \) which have a zero of order greater than or equal to \( n \) at \( (0, 1 - n^{-3}) \) for \( n = 1, 2, \ldots \). Douglas and Yang [12] showed that \( M \ominus (zM + wM) \) is finite dimensional, while \( M \ominus (zM + wM) \) is not a generating set of \( M \). They raised the question what \( \text{dim}(M \ominus (zM + wM)) \) is
equal to. It is easy to check that both $M$ and $zM + wM$ are AF–cosubmodules, and $Z(zM + wM) \setminus Z(M) = \{(0, 0)\}$, $\text{card}(Z(zM + wM) \setminus Z(M)) = 2$. Theorem 3.1 thus implies that $\text{dim}(M \ominus (zM + wM))$ is equal to 2.

**Example 3.** Let $M$ be a submodule of the Bergman module $L^2_a(D)$ over the disk algebra. By Aleman, Richter and Sunderberg’s work [13], we know that $M \ominus zM$ is a generating set for $M$. It is easy to see that $\text{dim}(M \ominus zM) \leq \text{rank}(M)$. One thus concludes that $\text{dim}(M \ominus zM) = \text{rank}(M)$. Therefore, for any natural number $n$, unlike the Hardy module $H^2(D)$, the Bergman module $L^2_a(D)$ has a submodule $M$ with rank $n$ [10]. Let $M$ be an AF–cosubmodule of $L^2_a(D)$ with $\text{rank}(M) < \infty$. This implies that $zM$ also is an AF–cosubmodule. It is easy to check that $\text{card}(Z(zM) \setminus Z(M)) = 1$, and hence $\text{dim}(M \ominus zM) = 1$. We conclude that $\text{rank}(M) = 1$. It follows that every AF–cosubmodule with finite rank is generated by a single function.

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**References**


