Homogeneous quasi-invariant subspaces of the Fock space

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Abstract
In this paper, we prove that two homogeneous quasi-invariant subspaces are similar only if they are equal. Moreover, we exhibit an example to show how to determine the similarity orbits of quasi-invariant subspaces.

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1 Introduction
Recall that the Hardy space $H^2(\mathbb{D})$ over the open unit disk $\mathbb{D}$ is the closed subspace of $L^2(\mathbb{T})$ spanned by the non-negative powers of the coordinate function $z$. If $M$ is a (closed) subspace of $H^2(\mathbb{D})$ that is invariant for the multiplication operator $M_z$, then Beurling’s theorem says that there exists an inner function $\eta$ such that $M = \eta H^2(\mathbb{D})$. Beurling’s theorem has played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, and let $X$ be a Hilbert space consisting of analytic functions in $\Omega$ such that $1 \in X$, and for each polynomial $p$ and each $h \in X$, $p h \in X$. If $M$ is a closed subspace of $X$ such that $p M \subseteq M$ for every polynomial $p$, we call that $M$ is an invariant subspace for the function space $X$.

However, despite the great development in these fields over the past fifty years, one can not understand invariant subspace lattices of function spaces

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better. Hence attention is directed to investigating equivalence classes of
invariant subspaces of function spaces under similarity or unitary equivalence.

Along this line, Axler, Agrawal, Bourdon, Douglas, Guo, Paulsen, Putinar, Salinas, etc. have done a lot of work, see [AB, AS, ACD, DP, DPSY, Guo1, Guo2, Guo3, Pu, Ric] and references there. The extension of the above
some results to the Fock space, and analytic Hilbert spaces on the complex
plane were considered by Guo and Zheng [GZh], and by Chen, Guo and Hou
[CGH].

The Fock space, is the analog of the Bergman space in the context of the
complex $n$-space $\mathbb{C}^n$. It is a Hilbert space consisting of entire functions in
$\mathbb{C}^n$. Let

$$d\mu(z) = e^{-|z|^2/2} dv(z)(2\pi)^{-n}$$

be the Gaussian measure on $\mathbb{C}^n$ ($dv$ is the ordinary Lebesgue measure). The
Fock space $L_2^a(\mathbb{C}^n, d\mu)$ (for short, $L_a(\mathbb{C}^n)$), by definition, is the space of all
$\mu$-square-integrable entire functions on $\mathbb{C}^n$. It is easy to see that $L_2^a(\mathbb{C}^n)$ is
a closed subspace of $L_2(\mathbb{C}^n)$ with the reproducing kernel functions
$K_\lambda(z) = e^{\lambda z/2}$, and the normalized reproducing kernel functions
$k_\lambda(z) = e^{\bar{\lambda}z/2} - |\lambda|^2/4$,

(here $\bar{\lambda}z = \sum_{i=1}^n \bar{\lambda}_i z_i$).

The next proposition tell one that there exist no nontrivial invariant
subspace for the Fock space. The proposition first appeared in [GZh]. We remark
it here for reader convenience.

**Proposition 1.1.** Let $M$ be a (closed) subspace of $L_2^a(\mathbb{C}^n)$, and $M \neq \{0\}$. If $f$

is an entire function on $\mathbb{C}^n$ such that $f M \subset M$, then $f$ is a constant.

**Proof.** By the assumption, the closed graph theorem implies that the
multiplication by $f$ on $M$, denoted by $M_f$, is a bounded operator. Use $\tilde{K}_\lambda$ to
denote the reproducing kernel functions associated with $M$, and $\tilde{k}_\lambda$ the
normalized reproducing kernel functions. Since

$$< f \tilde{k}_\lambda, \tilde{k}_\lambda > = f(\lambda),$$

we see that

$$|f(\lambda)| = | < f \tilde{k}_\lambda, \tilde{k}_\lambda > | \leq \|M_f\|,$$

and hence $f$ is a bounded entire function on $\mathbb{C}^n$. So, $f$ is a constant.

By Proposition 1.1, there exist no nontrivial invariant subspace for the
Fock space. Thus, an appropriate substitute for invariant subspace, the so-
called quasi-invariant subspace is needed. Namely, a (closed) subspace $M$ of
the Fock space is called quasi-invariant if the relation $pf \in L^2_a(\mathbb{C}^n)$ implies $pf \in M$ for any $f \in M$ and any polynomial $p$. Equivalently, $M$ is quasi-invariant if $pM \cap L^2_a(\mathbb{C}^n) \subset M$ for each polynomial $p$.

Let $M_1$ and $M_2$ be two quasi-invariant subspaces. We say that they are similar (unitarily equivalent) if there exists an invertible operator (a unitary operator) $A : M_1 \rightarrow M_2$ which satisfies that if $pf \in M_1$ (here $f \in M_1$, and $p$ is a polynomial), then $A(pf) = pA(f)$. Thus by the definition, the relation $pf \in M_1$ forces that $pA(f) \in M_2$. It is easy to see that similarity (unitary equivalence) is an equivalence relation in the category of all quasi-invariant subspaces.

It is well known that for each analytic function space $X$ on a bounded domain $\Omega$, the closure $[I]$ of an ideal $I$ of polynomial ring $\mathbb{C}$ is an invariant subspace of $X$. However it is never obvious if the closure $[I]$ of $I$ on the Fock space $L^2_a(\mathbb{C}^n)$ is quasi-invariant. In section 2, we first prove that the closure of a homogeneous ideal is quasi-invariant. Furthermore, it is shown that two homogeneous quasi-invariant subspaces are similar only if they are equal. In section 3, we determine the similarity orbit of the quasi-invariant subspace $[z^n]$. Namely, the similarity orbit of $[z^n]$ consists of $[p(z)]$, where $p(z)$ range over all polynomials in the variable $z$ with $\deg p = n$.

2 Homogeneous quasi-invariant subspaces of the Fock space

It is well known that for each analytic Hilbert space $X$ on a bounded domain $\Omega$, the closure $[I]$ of an ideal $I$ of polynomial ring $\mathbb{C}$ is an invariant subspace of $X$. However it is never obvious if the closure $[I]$ of $I$ on the Fock space $L^2_a(\mathbb{C}^n)$ is quasi-invariant.

Here we give a proposition which shows that the closure of a homogeneous ideal is quasi-invariant. Recall that an ideal $I$ is homogeneous if the relation $p \in I$ implies that all homogeneous components of $p$ are in $I$. Equivalently, an ideal $I$ is homogeneous if and only if $I$ is generated by homogeneous polynomials.

**Proposition 2.1.** Let $I$ be a homogeneous ideal. Then the closure $[I]$ of $I$ on the Fock space is quasi-invariant.

**Proof.** Let $f \in [I]$, and $f = \sum_{k=0}^{\infty} f_k$ be $f$'s homogeneous expression. We claim that every $f_k$ is in $I$. To prove the claim, we let $I_k$ consist of all those
\( p \in I \) with homogeneous degree of \( p \) being at most \( k \). Then \( I_k \) is of finite dimension. From the relation \( f \in [I] \), there is a sequence \( \{p_n\} \) in \( I \) such that \( p_n \to f \) as \( n \to \infty \). This implies that \( p_n^{(k)} \to f_k \), where \( p_n^{(k)} \) denote \( k \)-homogeneous component of \( p_n \). Since \( I \) is homogeneous, \( p_n^{(k)} \) belong to \( I \), and hence they are in \( I_k \). Because \( I_k \) is finitely dimensional, and hence closed, this forces that \( f_k \in I \).

Assume that \( qf \in L^2(\mathbb{C}^n) \) for some polynomial \( q \). Let \( q = \sum_{i=0}^m q_i \) be the homogeneous expression of \( q \). Then the homogeneous expression of \( qf \) is given by

\[
qf = \sum_{n=0}^{\infty} \left( \sum_{i+j=n} q_if_j \right).
\]

Now it is easy to derive that \( qf \in [I] \) by the above homogeneous expression of \( qf \). It follows that \( [I] \) is quasi-invariant.

**Theorem 2.2.** Let \( I_1 \) and \( I_2 \) be homogeneous ideals. Then \( [I_1] \) and \( [I_2] \) are similar if and only if \( I_1 = I_2 \).

To prove theorem, we need some preliminaries.

Let \( B_n \) be the unit ball of \( \mathbb{C}^n \), and \( \partial B_n \) be the boundary of \( B_n \). We let \( \sigma \) be the unique rotation-invariant positive Borel measure on \( \partial B_n \) for which \( \sigma(\partial B_n) = 1 \). As usual, \( H^2(B_n) \) denotes the Hardy space on the unit ball \( B_n \). Let \( M_1; M_2 \) be invariant subspaces of \( H^2(B_n) \). We call that a bounded linear operator \( A : M_1 \to M_2 \) is a module map if \( A(ph) = pA(h) \) for any polynomial \( p \) and each \( h \in M_1 \).

**Proposition 2.3.** If \( M_1 \) and \( M_2 \) are invariant subspaces of \( H^2(B_n) \) and \( A : M_1 \to M_2 \) is a module map, then there exists a bounded function \( \phi \) on \( \partial B_n \) such that \( A(h) = \phi h \) for any \( h \in M_1 \).

**Proof.** From Rudin [Ru2], we see that all inner functions on the \( B_n \) and their adjoints generate \( L^\infty(\partial B_n) \) in the weak*-topology. Set

\[
\mathcal{D} = \{ \bar{\eta}h : \eta \text{ are inner functions, and } h \in M_1 \}.
\]

Then \( \mathcal{D} \) is a dense linear subspace of \( L^2(\partial B_n) \). We define a map

\[
\hat{A} : \mathcal{D} \to L^2(\partial B_n)
\]

by

\[
\hat{A}(\bar{\eta}h) = \bar{\eta}A(h).
\]
Since $A$ is a module map, the above definition is well defined. From the relation
\[ \|\hat{A}(\bar{\eta}h)\| = \|A(h)\| \leq \|A\|\|\bar{\eta}h\|, \]
we see that $\hat{A}$ extends to a bounded map from $L^2(\partial B_n)$ to $L^2(\partial B_n)$. It is obvious that $\hat{A}$ satisfies
\[ \hat{A}M_g = M_gA \]
for any $g \in L^\infty(\partial B_n)$, and hence there exists a function $\phi \in L^\infty(\partial B_n)$ such that
\[ \hat{A} = M_\phi. \]
This insures that $A(h) = \phi h$ for any $h \in M_1$.

Let $A(B_n)$ be the ball algebra, that is, $A(B_n)$ consists of all functions $f$, which are analytic on $B_n$, and continuous on the closure $\overline{B_n}$ of $B_n$.

**Lemma 2.4.** Assume $n > 1$. Let $f, g \in A(B_n)$, and $|f(\xi)| \leq |g(\xi)|$ for each $\xi \in \partial B_n$, then also $|f(z)| \leq |g(z)|$ for every $z \in B_n$.

**Proof.** See [Ru1, Theorem 14.3.3].

For a polynomial $p$, if $p$ has its homogeneous expression $p = p_0 + p_1 + \cdots + p_l$ with $p_l \neq 0$, we call that $p$ is of homogeneous degree $l$, and denoted by $\text{deg}_h p$.

**Proposition 2.5.** Let both $\phi_1$ and $\phi_2$ be homogeneous polynomials, and $\text{deg}_h \phi_1 \geq \text{deg}_h \phi_2$. If there exists a constant $\gamma$ such that on the Fock space $L^2_a(\mathbb{C}^n)$,
\[ \|\phi_1 \phi\| \leq \gamma \|\phi_2 \phi\| \]
for any homogeneous polynomial $\phi$, then there is a constant $C$ such that $\phi_1 = C \phi_2$.

**Proof.** In the case $n = 1$, a straightforward verification shows that the conclusion is true. Below we may assume $n > 1$. First let us recall that integration in polar coordinates (corresponding the volume measure) is given by [Ru1, pp.13]
\[ \int_{C^n} f dv = \frac{2n\pi^n}{n!} \int_0^\infty \int_{\partial B_n} f(r\xi)d\sigma(\xi) dr. \]
Then by the inequality
\[ \|\phi_1 \phi\|^2 \leq \gamma^2 \|\phi_2 \phi\|^2, \]
one has that
\[ \|\phi_1\phi\| = \int_{C^n} |\phi_1|^2 |\phi|^2 e^{-\frac{|z|^2}{2}} dv \]
\[ = \frac{2n\pi^n}{n!} \int_0^\infty r^{2(n+k_1+l)-1} e^{-\frac{|z|^2}{2}} dr \int_{\partial B_n} |\phi_1(\xi)\phi(\xi)|^2 d\sigma(\xi) \]
\[ \leq \gamma^2 \int_{C^n} |\phi_2|^2 |\phi|^2 e^{-\frac{|z|^2}{2}} dv \]
\[ = \frac{2n\pi^n\gamma^2}{n!} \int_0^\infty r^{2(n+k_2+l)-1} e^{-\frac{|z|^2}{2}} dr \int_{\partial B_n} |\phi_2(\xi)\phi(\xi)|^2 d\sigma(\xi), \]

where \( k_1, k_2, l \) are the homogeneous degrees of homogeneous polynomials \( \phi_1, \phi_2, \phi \) respectively. From the formula
\[ \int_0^\infty r^{2m+1} e^{-\frac{|r|^2}{2}} dr = 2^m m! \]
and \( k_1 \geq k_2 \), we obtain that
\[ \int_{\partial B_n} |\phi_1(\xi)\phi(\xi)|^2 d\sigma(\xi) \leq \gamma^2 \int_{\partial B_n} |\phi_2(\xi)\phi(\xi)|^2 d\sigma(\xi). \]

Now let \( r \) be any polynomial with its homogeneous expression
\[ r = r_0 + r_1 + \cdots r_t. \]
Then on the Hardy space \( H^2(B_n) \),
\[ \phi_1 r_i \perp \phi_1 r_j; \quad \phi_1 r_i \perp \phi_1 r_j \]
if \( i \neq j \). This gives that
\[ \int_{\partial B_n} |\phi_1(\xi)r(\xi)|^2 d\sigma(\xi) = \int_{\partial B_n} |\sum_{i=0}^t \phi_1(\xi)r_i(\xi)|^2 d\sigma(\xi) \]
\[ = \sum_{i=0}^t \int_{\partial B_n} |\phi_1(\xi)r_i(\xi)|^2 d\sigma(\xi) \]
\[ \leq \gamma^2 \sum_{i=0}^t \int_{\partial B_n} |\phi_2(\xi)r_i(\xi)|^2 d\sigma(\xi) \]
\[ = \gamma^2 \int_{\partial B_n} |\phi_2(\xi)\sum_{i=0}^t r_i(\xi)|^2 d\sigma(\xi) \]
\[ = \gamma^2 \int_{\partial B_n} |\phi_2(\xi)r(\xi)|^2 d\sigma(\xi). \]
Let \([\phi_1]_n\) and \([\phi_2]_n\) be invariant subspaces of \(H^2(B_n)\) generated by \(\phi_1, \phi_2\) respectively. Then applying the preceding inequalities yields the following bounded module map

\[
B : [\phi_2]_n \to [\phi_1]_n, \quad B\phi_2r = \phi_1r.
\]

By Proposition 2.3, there is a bounded function \(f\) on \(\partial B_n\) such that \(B = M_f\). This implies that \(f\phi_2 = \phi_1\) on \(\partial B_n\). So,

\[
|\phi_1(\xi)| \leq \|f\|_{\infty}|\phi_2(\xi)|
\]

for every \(\xi \in \partial B_n\). By Lemma 2.4,

\[
|\phi_1(z)| \leq \|f\|_{\infty}|\phi_2(z)|
\]

for each \(z \in B_n\). Set

\[
\psi(z) = \phi_1(z)/\phi_2(z).
\]

Then \(\psi(z)\) is bounded on \(B_n\), and hence analytic. Since

\[
\phi_1(z) = \psi(z)\phi_2(z),
\]

and \(\phi_1, \phi_2\) are homogeneous, this yields that \(\psi(z)\) is homogeneous.

Now by the inequality \(\|\phi_1\phi\| \leq \gamma \|\phi_2\phi\|\) and the fact that on the Fock space,

\[
\phi_1 \psi_1 \perp \phi_1 \psi_2; \quad \phi_2 \psi_1 \perp \phi_2 \psi_2,
\]

for homogeneous polynomials \(\psi_1, \psi_2\) with \(\deg_h \psi_1 \neq \deg_h \psi_2\), we have that

\[
\|\phi_1r\| \leq \gamma \|\phi_2r\|
\]

for each polynomial \(r\), and hence for every entire function \(f\) with the property \(\phi_2f \in L^2_a(\mathbb{C}^n)\). This means that the relation \(\phi_2f \in L^2_a(\mathbb{C}^n)\) implies \(\phi_2 \psi f \in L^2_a(\mathbb{C}^n)\). It is not difficult to check that the quasi-invariant subspace \([\phi_2]\), the closure of \(\phi_2C\) on the Fock space, is given by

\[
[\phi_2] = \{\phi_2f \in L^2_a(\mathbb{C}^n) : f \in Hol(\mathbb{C}^n)\},
\]

and hence we can identify \([\phi_2]\) with the weighted Fock space \(L^2_a(\mathbb{C}^n, |\phi_2|^2d\mu)\) (here \(d\mu = e^{-\frac{|z|^2}{2}}dv\)). This means that for each \(\phi_2f \in [\phi_2]\),

\[
\|\phi_2f\| = \|f\|_2
\]
where \( \|f\|_2 \) is the norm of \( f \) in the space \( L^2_a(\mathbb{C}^n, |\phi_2|^2 d\mu) \).

By the closed graph theorem, the multiplication \( M_\psi \) defined by \( \psi \) is a bounded linear operator on the weighted Fock space \( L^2_a(\mathbb{C}^n, |\phi_2|^2 d\mu) \). It is easy to see that for every \( \lambda \in \mathbb{C}^n \setminus Z(\phi_2) \), the evaluation functional \( E_\lambda(f) = f(\lambda) \) is continuous on \( L^2_a(\mathbb{C}^n, |\phi_2|^2 d\mu) \). Let \( \hat{K}_\lambda \) be the reproducing kernel functions of \( L^2_a(\mathbb{C}^n, |\phi_2|^2 d\mu) \), and \( \hat{k}_\lambda \) be corresponding normalized reproducing kernel functions. Then one has that

\[
|< M_\psi \hat{k}_\lambda, \hat{k}_\lambda >| = |\psi(\lambda)| \leq \|M_\psi\|
\]

for each \( \lambda \in \mathbb{C}^n \setminus Z(\phi_2) \). This insures that

\[
|\psi(\lambda)| \leq \|M_\psi\|
\]

for each \( \lambda \in \mathbb{C}^n \). So, \( \psi \) is a constant. This completes the proof.

**Proof of Theorem 2.2.** Let \( A : [I_1] \to [I_2] \) be a similarity. For a homogeneous polynomial \( p \) in \( I_1 \), set \( q = A(p) \). If \( q = \sum_{i=0}^\infty q_i \) is the homogeneous expression of \( q \), then

\[
\|q\phi\|^2 = \sum_{i=0}^\infty \|q_i\phi\|^2 \leq \|A\|^2 \|p\phi\|^2
\]

for every homogeneous polynomial \( \phi \). We thus have that

\[
\|q_i\phi\|^2 \leq \|A\|^2 \|p\phi\|^2,
\]

for \( i \geq \deg_{h} p \). Applying Proposition 2.5 gives that \( q_i = \gamma p \) if \( i = \deg_{h} p \), \( q_i = 0 \) if \( i > \deg_{h} p \). From the proof of Proposition 2.1, we see that \( q = q_0 + q_1 + \cdots + q_{k-1} + \gamma p \) is in \( I_2 \), here \( k = \deg_{h} p \). First we claim that the constant \( \gamma \) is never zero. In fact, since \( I_2 \) is homogeneous, this means that \( q_i \) are in \( I_2 \) for \( i = 1, 2, \cdots k-1 \). Based on the same reasoning as above, one has that \( A^{-1}(q_0 + q_1 + \cdots + q_{k-1}) \) is a polynomial, and its homogeneous degree is at most \( k-1 \). So,

\[
p - A^{-1}(q_0 + q_1 + \cdots + q_{k-1}) \neq 0.
\]

This insures that \( \gamma \neq 0 \). According to the claim, \( I_1 \subset I_2 \). The same reasoning shows that \( I_2 \subset I_1 \), and hence \( I_1 = I_2 \).

Before ending this section let us remark the following proposition. Let \( M_1, M_2 \) be quasi-invariant, and let \( A : M_1 \to M_2 \) be a bounded linear map.
We call that \( A : M_1 \to M_2 \) is a quasi-module map if \( A(pf) = pA(f) \) whenever \( pf \in M_1 \) (here \( f \in M_1 \) and \( p \) is any polynomial). Moreover, for a quasi-invariant subspace \( M \), it is easy to verify that \( M \cap C \) is an ideal.

**Proposition 2.6.** Let \( A : M_1 \to M_2 \) be a quasi-module map. Then \( A \) maps \( M_1 \cap C \) to \( M_2 \cap C \).

**Proof.** We may assume that \( M_1 \) contains a nonzero polynomial \( p \). Set \( q = A(p) \). We claim that \( \deg_i q \leq \deg_i p \) for \( i = 1, 2, \cdots, n \), where \( \deg_i f \) denotes degree of an entire function \( f \) in the variable \( z_i \) (allowed as \( \infty \)).

Suppose that there is some \( i \), say, 1, such that \( \deg_1 q > \deg_1 p \). Then we expand \( p \) and \( q \) in the variable \( z_1 \) by

\[
p = p_0 + p_1 z_1 + \cdots + p_l z_1^l, \quad q = q_0 + q_1 z_1 + \cdots.
\]

Since \( \deg_1 q > \deg_1 p \), there exists a positive integer number \( s(> l) \) such that \( q_s \neq 0 \). From the equality

\[
||A(z_1^k p)||^2 = ||z_1^k q||^2 = \sum_{i=0}^{\infty} ||z_1^{k+i} q_i||^2,
\]

we have that

\[
||z_1^{k+s} q_s||^2 \leq ||A||^2 \sum_{i=0}^{l} ||z_1^{k+i} p_i||^2.
\]

Since

\[
||z_1^{k+s} q_s||^2 = ||z_1^{k+s}||^2 ||q_s||^2 = 2^{k+s}(k+s)! ||q_s||^2
\]

and

\[
||z_1^{k+i} p_i||^2 = 2^{k+i}(k+i)! ||p_i||^2, \quad i = 0, 1, 2, \cdots, l,
\]

for any natural number \( k \), we get that \( q_s = 0 \). This yields the desired contradiction, and hence Proposition 2.6 follows.

We endow the ring \( C \) with the topology induced by the Fock space \( L_a^2(C^n) \). For an ideal \( I \), we regard \( I \) as module over the ring \( C \).

**Corollary 2.7.** Let \( A : M_1 \to M_2 \) be a similarity. Then \( A \) induces a continuous module isomorphism from \( M_1 \cap C \) onto \( M_2 \cap C \).
3 The similarity orbit of \([z^n]\)

From Theorem 2.2, one sees that for homogeneous quasi-invariant subspaces, similarity only appears in the case of equality. Therefore, a natural problem is to determine the similarity orbit of quasi-invariant subspaces. Let \(M\) be a quasi-invariant subspace. Then the similarity orbit, \(\text{orb}_s(M)\), of \(M\) consists of all quasi-invariant subspaces which are similar to \(M\). There is no doubt that the problem is difficult. Here we will exhibit an example to show how to determine the similarity orbit.

For a polynomial \(p\), we let \([p]\) denote the closure of \(pC\) on the Fock space. Using sheaf theory or Theorem 2.3 in [Guo3], one easily verifies that for each \(g \in [p]\), there exists an entire function \(f\) such that \(g = pf\). Moreover, if \(p\) is homogeneous, then \([p]\) is quasi-invariant.

**Theorem 3.1.** On the Fock space \(L^2_a(C^2)\), the similarity orbit \(\text{orb}_s([z^n])\) of \([z^n]\) consists of \([p(z)]\), where \(p(z)\) range over all polynomials in the variable \(z\) with \(\deg p = n\).

**Proof.** We first claim: if \((z + \alpha)f \in L^2_a(C^2)\), then \(zf; f \in L^2_a(C^2)\), where \(\alpha\) is a constant. Let

\[
f = \sum_{n \geq 0} f_n(z)w^n
\]

be the expansion of \(f\) relative to the variable \(w\). Then

\[
(z + \alpha)f = \sum_{n \geq 0} (z + \alpha)f_n(z)w^n.
\]

By an easy verification, there exists positive constants \(C_\alpha, C'_\alpha\) such that

\[
C_\alpha \| zg(z) \| \leq \|(z + \alpha)g(z)\| \leq C'_\alpha \| zg(z) \|,
\]

for any entire function \(g(z)\) on the complex \(\mathbb{C}\). Since

\[
\|(z + \alpha)f\|^2 = \sum_{n \geq 0} \|(z + \alpha)f_n(z)\|^2 \|w^n\|^2 < \infty,
\]

this gives that

\[
\sum_{n \geq 0} \|zf_n\|^2 \|w^n\|^2 < \infty
\]
and hence $zf \in L^2_{a}(\mathbb{C}^2)$. Also from the relation $zf \in L^2_{a}(\mathbb{C}^2)$, one easily gets $f \in L^2_{a}(\mathbb{C}^2)$ and hence the claim follows.

Combining the above claim with induction, we see that for a polynomial $p(z)$ in the variable $z$, the relation $pf \in L^2_{a}(\mathbb{C}^2)$ implies $z^n f; z^{n-1} f; \cdots; f \in L^2_{a}(\mathbb{C}^2)$, here $n = \deg p$. The same reasoning shows that the relation $z^n f \in L^2_{a}(\mathbb{C}^2)$ implies $p(z) f \in L^2_{a}(\mathbb{C}^2)$, here $\deg p = n$.

Let $p(z)$ be a polynomial in the variable $z$ with $\deg p = n$. Then we can establish an inequality

$$C_1 \| z^n f \| \leq \| p(z) f \| \leq C_2 \| z^n f \|,$$

where $C_1; C_2$ are positive constants only depending on $p(z)$.

In fact, Let

$$f = \sum_{n \geq 0} f_n(z) w^n$$

be the expansion of $f$ relative to the variable $w$. Then

$$p(z) f = \sum_{n \geq 0} p(z) f_n(z) w^n.$$

It is easy to prove that there exists positive constants $C_1; C_2$, which depend only on $p(z)$ such that

$$C_1 \| z^n g(z) \| \leq \| p(z) g(z) \| \leq C_2 \| z^n g(z) \|,$$

for any entire function $g(z)$ on the complex plane. Now by the equality

$$\| p(z) f \|^2 = \sum_{n \geq 0} \| p(z) f_n(z) \|^2 \| w^n \|^2,$$

we thus have the inequality

$$C_1 \| z^n f \| \leq \| p(z) f \| \leq C_2 \| z^n f \|.$$

Since the homogeneous quasi-invariant subspace $[z^n]$ is given by

$$[z^n] = \{ z^n f \in L^2_{a}(\mathbb{C}^2) \mid f \text{ are entire functions} \},$$

The above inequality gives that

$$[p(z)] = \{ p(z) f \in L^2_{a}(\mathbb{C}^2) \mid f \text{ are entire functions} \},$$

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and hence \([p(z)]\) is quasi-invariant.

Now we establish a map

\[ A : [z^n] \rightarrow [p(z)], \quad z^n f \mapsto p(z)f. \]

Then by the preceding discussion and the closed graph theorem, \(A\) is continuous. Obviously, \(A\) is injective, surjective, and is a quasi-module map. So, \(A\) is a similarity.

On the other hand, we let \(M\) be quasi-invariant, and \(A : [z^n] \rightarrow M\) be a similarity. Set \(q = A(z^n)\). We claim that \(q\) is a polynomial in the variable \(z\), and \(\deg q = n\). To prove the claim, we expand \(q\) relative to the variables \(w\), by

\[ q = q_0(z) + wq_1(z) + w^2q_2(z) \cdots . \]

Assume that \(\deg_w q > \deg_w z^n = 0\), here \(\deg_w q\) denotes degree of \(q\) in the variable \(w\) (allowed as \(\infty\)). Then there exists a positive integer \(s\) such that \(q_s(z) \neq 0\). Since

\[ \|A(w^k z^n)\|^2 = \|w^k q\|^2 = \sum_{i=0}^{\infty} \|w^{k+i} q_i(z)\|^2, \]

this implies that

\[ \|w^{k+s} q_s(z)\|^2 \leq \|A\|^2 \|w^k z^n\|^2 \]

for any positive integer \(k\). Since

\[ \|w^{k+s} q_s(z)\|^2 = 2^{k+s} (k+s)! \|q_s(z)\|^2 \quad \text{and} \quad \|w^k z^n\|^2 = 2^{k+n} k! n! \]

for any positive integer \(k\), it is easy to see that \(q_s = 0\). This contradicts the assumption, and hence \(\deg_w q = 0\). So, \(q\) depends only on the variable \(z\).

Now we expand \(q\) in the variable \(z\) by

\[ q(z) = \sum_{m \geq 0} a_m z^m. \]

If there is a positive integer \(l\), and \(l > n\) such that \(a_l \neq 0\), then the equality

\[ A(z^k z^n) = \sum_{m \geq 0} a_m z^{k+m} \]

implies that

\[ |a_l|^2 \|z^{k+l}\|^2 \leq \|A\|^2 \|z^{k+n}\|^2. \]
This leads to the following
\[ 2^{k+i} |a_l|^2 (k + l)! \leq 2^{k+n} \|A\|^2 (k + n)! \]
for any positive integer \( k \). This clearly is impossible, and hence \( q(z) \) is a polynomial in the variable \( z \) with deg \( q \leq n \).

Set \( s = \text{deg} \, q \). Then \( M = [q] \) is similar to \([z^s]\) by preceding discussion, and hence \([z^s]\) is similar to \([z^n]\). An easy verification shows \( s = n \). This means that \( q(z) \) is a polynomial in the variable \( z \) with degree \( n \).

Based on the above discussion, we conclude that the similarity orbit \( \text{orb}_z([z^n]) \) of \([z^n]\) consists of \([p(z)]\), where \( p(z) \) range over all polynomials in the variable \( z \) with \( \text{deg} \, p = n \).

**Remark 3.2.** From the proof of Theorem 3.1, it is not difficult to see that Theorem 3.1 remains true in the case of the Fock space \( L^2_a(\mathbb{C}^n) \).

**References**


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