Defect operators for submodules of $H^2_d$

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Abstract. This paper mainly concerns defect operators for submodules of the Hilbert module $H^2_d$ on the unit ball. The defect operator carries key information about operator theory and structure of submodules. For graded submodules we investigate when such submodules belong to Schatten class $L^p$. Furthermore, it is shown that a submodule of $H^2_d$ with finite rank has necessarily finite codimension. This implies an affirmative solution for a problem of Arveson.

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Introduction

In the study of multivariable operator theory, there is a natural approach via Hilbert module [DP]. Let $T = (T_1, \ldots, T_d)$ be a tuple of commuting operators acting on a Hilbert space $H$. Then, one naturally makes $H$ into a Hilbert module over the polynomial ring $C[z_1, \ldots, z_d]$. The $C[z_1, \ldots, z_d]$-module structure is defined by

$$p \cdot \xi = p(T_1, \ldots, T_d)\xi, \quad p \in C[z_1, \ldots, z_d], \quad \xi \in H.$$ 

Following Arveson’s language [Arv1], [Arv2], a Hilbert module is said to be contractive if the following condition is satisfied:

$$\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2, \quad \xi_1, \ldots, \xi_d \in H$$

that is, $T_1T_1^* + \cdots + T_dT_d^* \leq 1$. A such commuting $d$-tuple $(T_1, \ldots, T_d)$ is called a $d$-contraction and the rank of a contractive Hilbert module is defined as the rank of its defect operator $\Delta(H) = (I - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}$. The rank of $H$ is written $\operatorname{rank}(H)$. We say:

1) $H$ is a finite rank module if $\operatorname{rank}(H) < \infty$.

2) $H$ is a Hilbert-Schmidt module if $\Delta(H)$ is a Hilbert Schmidt operator.

3) $H$ is a Schatten class $L^p$ module if $\Delta(H) \in L^p$.

It is well known that in the theory of a single contraction operator, the shift operator plays a basic role [NF]. In the dimension $d \geq 2$, there is the $d$-dimensional counterpart of
the shift, the so-called $d$-shift. The $d$-shift acts naturally on a space of analytic functions defined on the open unit ball $B_d \subset \mathbb{C}^d$. This space plays an important role in the theory of contractive Hilbert modules, as shown by Arveson [Arv1], [Arv2], [Arv3]. Based on this function space, many of the operator-theoretic aspects of function theory in the dimension $d = 1$ generalize to higher dimensions. Recall that this function space, denoted by $H^2_{d}$, defined on the unit ball $B_d$ is derived by the reproducing kernel

$$K_{\lambda}(z) = \frac{1}{1 - \langle z, \lambda \rangle},$$

where $\langle z, \lambda \rangle = \sum_{j=1}^{d} z_j \bar{\lambda}_j$. The $d$-shift is the tuple of the multiplication operators $\{M_{z_1}, \ldots, M_{z_d}\}$ acting on $H^2_{d}$ by the coordinate functions.

The space $H^2_{d}$, known as the symmetric Fock space, has been comprehensively studied by Arveson from the point of view of Hilbert modules. We refer the reader to the references [Arv1], [Arv2], [Arv3] for a far-reaching operator-theoretic and operator-algebraic developments of the theory of contractive Hilbert modules. The space $H^2_{d}$ is an important reproducing kernel function space; as shown in [AM1], [AM2], whose kernel is a universal (in an appropriate sense) complete Nevanlinna-Pick kernel. It was studied also in [BTV], [EP] from the point of view of multiplier spaces and [Fa2], [Gr] from the points of view of geometry and homology, respectively. The paper [PPS] also concerns Bohr’s inequality on $H^2_{d}$.

From the point of view of Hilbert modules, the space $H^2_{d}$ admits a natural $C[z_1, \ldots, z_d]$-module structure coming from multiplication by polynomials. Then, as shown in [Arv2], the module $H^2_{d}$ plays the role of universal objects in the category of pure contractive Hilbert modules. Therefore, to study pure contractive Hilbert modules, one should focus attention on the module $H^2_{d}$ and its submodules. Concerning submodules of $H^2_{d}$, a natural problem was asked by Arveson [Arv2], [Arv4].

**Problem.** In dimension $d \geq 2$, must the rank of each nonzero submodule of $H^2_{d}$, which has infinite codimension in $H^2_{d}$, be infinite?

In the present paper, we will give an affirmative answer to the above Problem. We will also show that a defect operator carries key information about operator theory and structure of submodules. Section 1 collects some basic material associated with the module $H^2_{d}$. A notion of defect function for a submodule is introduced, and one will see that there is a natural connection between defect function and defect operator. As we will see in Section 1, a submodule is uniquely determined by its defect function, and hence is uniquely determined by its defect operator. This says that the defect (function) operator is a complete invariant for submodules in the sense of (function) operator theory. Section 2 mainly considers when a graded submodule belongs to Schatten class $L^p$. It is shown that a graded submodule belongs to the Schatten class $L^p$ for some $p \leq 2$ only if the submodule is finite codimensional. Moreover, if $M$ is a graded submodule of $H^2_{d}$, then the submodule belongs to every Schatten class $L^p$ for $p > 2$. We give examples to show that this result is sharp in an appropriate sense. In Section 3, we attack the Arveson’s problem mentioned above. Combining function theory in the unit ball, the theory of ideals of polynomials and some techniques in Fang’s paper [Fa1] gives an affirmative answer to this problem. By Arveson’s
dilation theorem this means that each nonzero submodule with infinite codimension is never unitarily equivalent to a quotient of any finite rank free Hilbert module. In Section 4 we consider the pure contractive Hilbert module \( \mathcal{H}_v^d (v > 0) \) on the unit ball defined by the kernel \( K_v^d (z) = 1/(1-\langle z, \lambda \rangle)^v \). It is shown that a submodule \( M \) of \( \mathcal{H}_v^d (d \geq 2) \) satisfies \( \text{rank}[P_M, M_j] < \infty \) for \( j = 1, \ldots, d \) if and only if it has finite codimension.

1. Preliminaries

By the classical Beurling’s theorem [Beu], each submodule \( M \) of the Hardy module \( H^2(D) \) on the unit disk \( D \) has the form \( M = \eta H^2(D) \), where \( \eta \) is an inner function. This means that the orthogonal projection \( P_M \) from \( H^2(D) \) onto \( M \) can be written as:

\[
P_M = M \eta M^{\eta^*}.
\]

Letting \( K_\lambda \) and \( K_\lambda^M \) be the reproducing kernels of \( H^2(D) \) and \( M \), respectively, then

\[
K_\lambda^M = P_M K_\lambda = \bar{\eta(\lambda) \eta} K_\lambda,
\]

and hence

\[
|\eta(\lambda)|^2 = \|K_\lambda^M\|^2 / \|K_\lambda\|^2 = \|P_M K_\lambda\|^2,
\]

where \( k_\lambda = K_\lambda / \|K_\lambda\| \) is the normalized reproducing kernel of \( H^2(D) \). It follows that the inner function can be captured from (1.1).

Motivated by (1.1), we will introduce a notion of defect function for a submodule \( M \) of the module \( H_d^2 \). By a submodule \( M \) of \( H_d^2 \) we mean that \( M \) is closed, and \( M \) is invariant under multiplication by polynomials. Defect function \( D_M(\lambda) \) for the submodule \( M \) is defined as follows:

\[
(1.2) \quad D_M(\lambda) = \|K_\lambda^M\| / \|K_\lambda\| = \|P_M K_\lambda\| / \|K_\lambda\| = \|P_M k_\lambda\|, \quad \lambda \in \mathbb{D}_d,
\]

where \( K_\lambda^M = P_M K_\lambda \) be the reproducing kernel of the submodule \( M \), \( P_M \) is the orthogonal projection from \( H_d^2 \) onto \( M \) and \( k_\lambda = K_\lambda / \|K_\lambda\| \) is the normalized reproducing kernel of \( H_d^2 \). From the definition of defect function \( D_M \), we have

\[
D_M^2(\lambda) = \|P_M K_\lambda\|^2 / \|K_\lambda\|^2 = (1 - |\lambda_1|^2 - |\lambda_2|^2 - \cdots - |\lambda_d|^2)\|P_M k_\lambda\|^2
\]

\[
= (1 - |\lambda_1|^2 - |\lambda_2|^2 - \cdots - |\lambda_d|^2)\langle P_M K_\lambda, K_\lambda \rangle
\]

\[
= \langle (P_M - M_{z_1} P_M M_{z_1}^* - M_{z_2} P_M M_{z_2}^* - \cdots - M_{z_d} P_M M_{z_d}^*) K_\lambda, K_\lambda \rangle.
\]

By [Arv2], each submodule is pure contractive Hilbert module, which in turn implies

\[
P_M - M_{z_1} P_M M_{z_1}^* - M_{z_2} P_M M_{z_2}^* - \cdots - M_{z_d} P_M M_{z_d}^* \geq 0.
\]

The defect operator of the submodule \( M \) is defined as
\[ \Delta(M) = [P_M - M_{z_1} P_M M_{z_1}^* - M_{z_2} P_M M_{z_2}^* - \cdots - M_{z_d} P_M M_{z_d}^*]^{1/2}, \]

and the rank of the submodule \( M \) is defined by

\[ \text{rank}(M) = \text{rank} \Delta(M). \]

From the above expression, one sees that the defect function and the defect operator for the submodule \( M \) are connected by the relation

\[ D^2_M(\lambda) = \langle \Delta^2(M) K_\lambda, K_\lambda \rangle, \]

and the defect operator and the reproducing kernel of the submodule is connected by

\[ K^M_\lambda = P_M K_\lambda = K_\lambda \Delta^2(M) K_\lambda. \]

When \( d = 1 \), \( H^2_d \) is the classical Hardy module \( H^2(\mathbb{D}) \), and in this case there is an inner function \( \eta \) such that \( P_M = M_\eta M_\eta^* \), and hence

\[ \Delta^2(M) = P_M - M_z P_M M_z^* = \eta \otimes \eta, \]

where \( f \otimes g \) denotes rank one operator defined by \( f \otimes g(h) = \langle h, g \rangle f \). This shows that in the case of the classical Hardy module, inner functions can be recovered by defect operators.

The next proposition shows that a submodule is uniquely determined by its defect function, and hence is uniquely determined by its defect operator.

**Proposition 1.1.** Given submodules \( M \) and \( N \) of \( H^2_d \), if \( D_M(\lambda) = D_N(\lambda) \) for \( \lambda \in \mathbb{D}_d \), then \( M = N \). Therefore, if \( \Delta(M) = \Delta(N) \), then \( M = N \).

To prove the proposition we need the following lemma. The proof of the lemma appeared in [Eng]. Of course, the lemma can also be proved by using Taylor expansion.

**Lemma 1.2.** Let \( \Omega \) be a bounded complete Reinhardt domain (i.e. a bounded domain with the property that for \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \Omega \), if \( |\mu_i| \leq 1 \), \( i = 1, 2, \ldots, d \), then \( (\mu_1 \lambda_1, \mu_2 \lambda_2, \ldots, \mu_d \lambda_d) \in \Omega \)). Suppose \( f(\lambda, z) \) is a function on \( \Omega \times \Omega \), and it is analytic in \( z \), and co-analytic in \( \lambda \). If \( f(\lambda, \lambda) = 0 \) for any \( \lambda \in \Omega \), then \( f = 0 \).

**Proof of Proposition 1.1.** To obtain the desired conclusion, considering functions

\[ G_M(\lambda, z) = \langle \Delta^2(M) K_\lambda, K_z \rangle, \quad G_N(\lambda, z) = \langle \Delta^2(N) K_\lambda, K_z \rangle, \]

then \( G_M(\lambda, z) \) and \( G_N(\lambda, z) \) are analytic in \( z \), and co-analytic in \( \lambda \), respectively. By (1.3)

\[ G_M(\lambda, \lambda) = D^2_M(\lambda) = D^2_N(\lambda) = G_N(\lambda, \lambda), \]

applying Lemma 1.2 shows

\[ G_M(\lambda, z) = G_N(\lambda, z), \]
and hence $\Delta(M) = \Delta(N)$. Since

$$
\begin{align*}
(1 - \bar{z}_1 z_1 - \cdots - \bar{z}_d z_d) \langle P_M K_\lambda, K_\mu \rangle &= \langle \Delta^2(M) K_\lambda, K_\mu \rangle = \langle \Delta^2(N) K_\lambda, K_\mu \rangle \\
&= (1 - \bar{z}_1 z_1 - \cdots - \bar{z}_d z_d) \langle P_N K_\lambda, K_\mu \rangle,
\end{align*}
$$

we see $M = N$.

Now let us recall some basic results from [Arv2]. Arveson proved that there exists a multiplier sequence $\{\phi_n\}$ of $H^2_d$ such that

$$
(1.5) \quad P_M = \sum_{n \geq 1} M_{\phi_n} M_{\phi_n}^* \text{ (SOT)},
$$

and hence

$$
(1) \quad \Delta^2(M) = \sum_{n \geq 1} \phi_n \otimes \phi_n \text{ (SOT)};
$$

$$
(2) \quad D_M^2(\lambda) = \| P_M k_\lambda \|^2 = \sum_{n \geq 1} |\phi_n(\lambda)|^2, \quad \lambda \in \mathbb{D}_d.
$$

Note that the above term (1) comes from the following reasoning. Indeed, for each multiplier $\phi$ of $H^2_d$, we have

$$
M_{\phi}M_{\phi}^* - M_{z_1}M_{\phi}M_{z_1}^* - \cdots - M_{z_d}M_{\phi}M_{z_d}^*
$$

$$
= M_{\phi}(I - M_{z_1}M_{z_1}^* - \cdots - M_{z_d}M_{z_d}^*)M_{\phi}^*
$$

$$
= M_{\phi}(1 \otimes 1)M_{\phi}^* = \phi \otimes \phi \quad \text{(by the next Lemma 1.3)}
$$

and the term (2) is from (1.3).

Moreover, if $M$ contains a nonzero polynomial, Arveson [Arv2] proved that $\{\phi_n\}$ is an inner sequence, i.e. for almost all $z \in \partial \mathbb{D}_d$ with respect to the normalized natural measure $\sigma$ on $\partial \mathbb{D}_d$,

$$
\sum_{n \geq 1} |\phi_n(\lambda)|^2 \to 1
$$

as $\lambda \to z$ non-tangentially. The general case was proved by Greene, Richter and Sundberg [GRS], that is, for each submodule $M$, any sequence satisfying (1.5) is inner. Furthermore, by [Kr], Corollary 2.1.7, we see that the defect function $D_M(\lambda)$ is subharmonic, and $D_M(\lambda) = 1$ almost everywhere $d\sigma$ on $\partial \mathbb{D}_d$.

Let $H_n$ denote the space of all homogeneous polynomials with degree $n$. Define the number operator $N$ acting on $H^2_d$ as follows:

$$
N = \sum_{n \geq 0} nP_{H_n},
$$
where \( P_{H_n} \) is the orthogonal projection from \( H^2_2 \) onto \( H_n \). The following lemma will be used in this paper (cf. [Arv1]).

**Lemma 1.3.** We have

1. \( M_{z_1}^* M_{z_1}^* + \cdots + M_{z_d}^* M_{z_d}^* = I - 1 \otimes 1; \)

2. \( M_{z_1}^* M_{z_1} + \cdots + M_{z_d}^* M_{z_d} = (d + N)(1 + N)^{-1}. \)

### 2. Graded submodules

As we have seen, if \( M \) is a submodule of the classical Hardy module \( H^2(D) \), then \( M \) has rank one. It has been argued that \( H^2_2 (d \geq 2) \) is perhaps the ‘correct’ generalization to multivariables of the classical Hardy module. Therefore, in dimension \( d \geq 2 \), one might ask if for any submodule \( M \) of \( H^2_2 \), \( M \) is of finite rank. Clearly, if \( M \) is of finite codimension in \( H^2_2 \), then \( M \) is of finite rank. Conversely, Arveson [Arv2] proved that for a graded submodule \( M \), \( M \) is of finite rank only if \( M \) is finite codimensional. This section will mainly consider when a graded submodule \( M \) belongs to Schatten class \( \mathcal{L}^p \). Before continuing let us see a result of Arveson.

**Example 1.** It is easy to see that a submodule is graded (in the sense of [Arv2]) if and only if the submodule is generated by homogeneous polynomials, and in this case, by Hilbert’s basis theorem [ZS], the submodule is generated by finitely many homogeneous polynomials. In [Arv2], Arveson proved that for every graded submodule \( M \), if \( M \) has finite rank, then \( M \) is of finite codimension in \( H^2_2 \). In fact, if \( M \) is graded, then \( M \) has an orthonormal basis consisting of homogeneous polynomials, and hence \( P_M \) maps polynomials to polynomials. This implies that the operator \( \Delta^2(M) \) maps polynomials to polynomials. Assume rank \( \Delta(M) = l < \infty \). By rank \( \Delta^2(M) = \text{rank} \Delta(M) \), there exist polynomials \( p_1, \ldots, p_l \) such that

\[
\Delta^2(M) = \sum_{k=1}^{l} p_k \otimes p_k.
\]

Since

\[
K_j \Delta^2(M) K_i = \sum_{k=1}^{l} M_{p_k} M_{p_k}^* K_i,
\]

using (1.4) yields

\[
P_M = \sum_{k=1}^{l} M_{p_k} M_{p_k}^*,
\]

and hence

\[
P_{H^2_2 \otimes M} = I - \sum_{k=1}^{l} M_{p_k} M_{p_k}^*.
\]
Noticing that \( \{ p_1, p_2, \ldots, p_l \} \) is a finite polynomial inner sequence, applying Theorem 5.7 in [Arv1] shows that the projection \( P_{H^2_d} \ominus M \) is compact, and hence \( M \) is of finite codimension in \( H^2_d \).

We can generalize Example 1 to the following version.

**Theorem 2.1.** Let \( M \) be a graded submodule of \( H^2_d \). If \( M \) is Hilbert-Schmidt module, then \( M \) is finite codimensional in \( H^2_d \).

As an immediate consequence, we have

**Corollary 2.2.** Let \( M \) be a graded submodule of \( H^2_d \). If \( M \) belongs to Schatten class \( \mathcal{L}^p \) for some \( p \leq 2 \), then \( M \) is finite codimensional in \( H^2_d \).

Here we give an example to show that Corollary 2.2 is sharp in an appropriate sense.

**Example 2.** Considering the submodule \( M \) generated by \( z_1, z_2, \ldots, z_d-1 \), then \( \dim H^2_d / M = \infty \). We will show that \( \Delta(M) \) belongs to every Schatten class \( \mathcal{L}^p \) for \( p > 2 \). In fact, it is easy to check that the reproducing kernel of \( H^2_d \ominus M \) is given by

\[
K^H^2_d \ominus M = \frac{1}{1 - \lambda_d z_d},
\]

and hence

\[
K^M = \frac{1}{1 - \lambda_1 z_1 - \cdots - \lambda_d z_d} - \frac{1}{1 - \lambda_d z_d}.
\]

By (1.4) we have

\[
\Delta^2(M) K = 1 - \frac{\lambda_1 z_1 + \cdots + \lambda_d z_d}{1 - \lambda_d z_d} = \sum_{n=0}^{\infty} \sum_{i=1}^{d-1} \lambda_i \lambda_{d-i} z_i z_d^n \]

\[
= \left[ \sum_{n=0}^{\infty} \sum_{i=1}^{d-1} z_i z_d^n \otimes z_i z_d^n \right] K.
\]

This shows

\[
\Delta^2(M) = \sum_{n=0}^{\infty} \sum_{i=1}^{d-1} z_i z_d^n \otimes z_i z_d^n.
\]

Setting \( e_{in} = z_i z_d^n / \| z_i z_d^n \| \), then \( e_{in} \perp e_{jm} \) if \( (i, n) \neq (j, m) \). Since \( \| z_i z_d^n \|^2 = 1 / (n + 1) \) for every integer \( n \geq 0 \), we obtain

\[
\Delta^2(M) = \sum_{n=0}^{\infty} \sum_{i=1}^{d-1} \frac{1}{n + 1} e_{in} \otimes e_{in}.
\]
This means that $\Delta(M)$ belongs to every Schatten class $\mathcal{L}^p$ for $p > 2$, but $M$ is not a Hilbert-Schmidt module.

**Remark.** Denoting the class of all graded Schatten class $\mathcal{L}^p$ submodules by $\mathcal{G}^d_p$, and the class of all finite rank graded submodules by $\mathcal{G}^d_0$, and the class of all graded submodules with finite codimensions by $\mathcal{G}^d_{\infty}$, and the class of all graded submodules by $\mathcal{G}^d_y$, then $\mathcal{G}^d_p \subseteq \mathcal{G}^d_q$ if $p < q$. By Corollary 2.2 and the above example, we have

$$
\mathcal{G}^d_{\infty} = \mathcal{G}^d_0 = \mathcal{G}^d_2 \subseteq \mathcal{G}^d_p \subseteq \mathcal{G}^d_{\infty}, \quad p > 2.
$$

Moreover, it is easy to see that if two submodules are unitarily equivalent, then their defect operators are necessarily unitarily equivalent. Therefore, the submodule in Example 2 is not unitarily equivalent to any submodule with finite codimension. This is highly distinct from the case of dimension one since any two submodules of $H^2(\mathbb{D})$ are unitarily equivalent. Concerning the equivalence problem for general analytic Hilbert modules, a comprehensive list of references can be found in [CG], [DP], [DPSY], [Guo1], [Guo2].

Before going on, recall that $H_n$ denotes the space of all homogeneous polynomials of degree $n$ for $n = 0, 1, 2, \ldots$. For a linear subspace $V_n$ of $H_n$, setting

$$
V_{n+1} = z_1 V_n + \cdots z_d V_n,
$$

then $V_{n+1} \subset H_{n+1}$. To prove Theorem 2.1, we need the following proposition which is interesting in itself.

**Proposition 2.3.** For a nonzero linear subspace $V_n$ of $H_n$, we have

$$
\dim V_{n+1} \geq \frac{d + n}{1 + n} \dim V_n
$$

with equality holding if and only if $V_n = H_n$.

**Proof.** Considering the graded submodule $[V_n]$ of $H^2_d$ generated by $V_n$, then

$$
[V_n] = V_n \oplus V_{n+1} \oplus V_{n+2} \oplus \cdots.
$$

So, we have

\[
\Delta^2([V_n]) = P_{[V_n]} - M_{z_1} P_{[V_n]} M_{z_1}^* - \cdots - M_{z_d} P_{[V_n]} M_{z_d}^* = \sum_{k=n}^\infty P_{V_k} - \sum_{k=n}^\infty \left[ \sum_{i=1}^d M_{z_i} P_{V_k} M_{z_i}^* \right] = P_{V_n} + \sum_{k=n}^\infty \left[ P_{V_{k+1}} - \sum_{i=1}^d M_{z_i} P_{V_k} M_{z_i}^* \right].
\]

Setting

$$
\Delta_n^2 = P_{H_{n+1}} \Delta^2([V_n]) P_{H_{n+1}},
$$


then by the above expression we see

\[ \Delta_n^2 = P_{n+1} - \sum_{i=1}^{d} M_{zi} P_{n} M_{zi}^*. \]

Taking an orthonormal basis \( \{ p_j^{(n)} \}_{j=1}^{\dim V_n} \) of \( V_n \), then we have

\[
\text{Trace} \left[ \sum_{i=1}^{d} M_{zi} P_{n} M_{zi}^* \right] = \sum_{j=1}^{\dim V_n} \text{Trace} \left[ \sum_{i=1}^{d} M_{zi} (p_j^{(n)} \otimes p_j^{(n)}) M_{zi}^* \right]
\]

\[
= \sum_{j=1}^{\dim V_n} \sum_{i=1}^{d} \| M_{zi} p_j^{(n)} \|^2
\]

\[
= \sum_{j=1}^{\dim V_n} \left( \sum_{i=1}^{d} M_{zi}^* M_{zi} \right) p_j^{(n)} p_j^{(n)} \right)
\]

\[
= \sum_{j=1}^{\dim V_n} \left( d + n \right) \frac{1}{1 + n} p_j^{(n)} \right)
\]

\[
= \frac{d + n}{1 + n} \dim V_n \quad \text{(by Lemma 1.3)}.
\]

Since \( \Delta_n^2 \geq 0 \) and

\[
\text{(2.1)} \quad \text{Trace} \Delta_n^2 = \dim V_{n+1} - \text{Trace} \left[ \sum_{i=1}^{d} M_{zi} P_{n} M_{zi}^* \right]
\]

\[
= \dim V_{n+1} - \frac{d + n}{1 + n} \dim V_n,
\]

this insures

\[
\dim V_{n+1} \geq \frac{d + n}{1 + n} \dim V_n.
\]

If \( V_n = H_n \), then \( V_{n+1} = H_{n+1} \), and hence by [Arv1],

\[
\dim V_n = \frac{(n + d - 1)!}{(d - 1)!n!}, \quad \dim V_{n+1} = \frac{(n + d)!}{(d - 1)!(n + 1)!}.
\]

This implies that the equality in Proposition 2.3 holds.

Now we assume that the equality in Proposition 2.3 holds. Then by (2.1), we see \( \text{Trace} \Delta_n^2 = 0 \), and hence \( \Delta_n^2 = 0 \) because \( \Delta_n^2 \geq 0 \). This implies that for any \( \xi \in V_{n+1} \), we have

\[
\| \xi \|^2 = \sum_{i=1}^{d} \| P_{n} M_{zi}^* \xi \|^2.
\]

By Lemma 1.3 (1),
This shows \( M_{z_i}^* \xi \in V_n \) for \( i = 1, \ldots, d \). Consequently, for \( i, j = 1, \ldots, d \) we have

\[
M_{z_i}^* M_{z_j}^* V_{n+1} \subseteq V_{n+1}.
\]

Notice that for each monomial \( z_1^{x_1} \cdots z_d^{x_d} \), an easy computing gives

\[
M_{z_i}^* z_1^{x_1} \cdots z_i^{x_i} \cdots z_d^{x_d} = 0
\]

if \( x_i = 0 \), and

\[
M_{z_i}^* z_1^{x_1} \cdots z_d^{x_d} = \frac{x_i}{x_1 + \cdots + x_d} z_1^{x_1} \cdots z_i^{x_i} \cdots z_d^{x_d}
\]

if \( x_i \neq 0 \). Picking a polynomial \( p = z_1^{x_1} \cdots z_d^{x_d} + \cdots \) in \( V_{n+1} \), then by the above observation we have

\[
(M_{z_1}^* M_{z_{d-1}}^*)^{x_1 + \cdots + x_{d-1}} \cdots (M_{z_1}^* M_{z_2}^*)^{x_1 + x_2} (M_{z_2}^* M_{z_1}^*)^{x_1} p = \beta z_{n+1}^{x_1},
\]

where \( \beta \) is a nonzero constant, and hence \( z_{n+1}^{x_1} \in V_{n+1} \). Now for any monomial \( z_1^{x_1} \cdots z_d^{x_d} \) in \( H_{n+1} \), a similar reasoning gives

\[
(M_{z_1}^* M_{z_2}^*)^{n+1-(x_d+\cdots+2x_1)} \cdots (M_{z_{d-1}}^* M_{z_{d-1}}^*)^{n+1-x_d-x_{d-1}} (M_{z_{d-1}}^* M_{z_d}^*)^{n+1-x_d} z_{n+1}^{x_1} = \gamma z_1^{x_1} \cdots z_d^{x_d},
\]

where \( \gamma \) is a nonzero constant. This implies \( V_{n+1} = H_{n+1} \). Notice that

\[
\dim H_n = \frac{(n + d - 1)!}{(d - 1)! n!}, \quad \dim H_{n+1} = \frac{(n + d)!}{(d - 1)! (n + 1)!}.
\]

Therefore, the equality \( \dim V_{n+1} = \frac{d + n}{1 + n} \dim V_n \) insures \( \dim V_n = \dim H_n \). This means \( V_n = H_n \), completing the proof.

**Proof of Theorem 2.1.** Since \( M \) is graded, this implies that \( M \) can be decomposed as

\[
M = V_0 \oplus V_1 \oplus \cdots \oplus V_n \oplus \cdots,
\]

where \( V_n \) is a linear subspace of \( H_n \) for \( n = 0, 1, 2, \ldots \). So, we have

\[
\Delta^2(M) = P_M - M_{z_1} P_M M_{z_1}^* - \cdots - M_{z_d} P_M M_{z_d}^*
\]

\[
= \sum_{n=0}^{\infty} P_{V_n} - \sum_{n=0}^{\infty} \left[ \sum_{i=1}^{d} M_{z_i} P_{V_n} M_{z_i}^* \right]
\]

\[
= P_{V_0} + \sum_{n=0}^{\infty} \left[ P_{V_{n+1}} - \sum_{i=1}^{d} M_{z_i} P_{V_n} M_{z_i}^* \right].
\]
As done in Proposition 2.3, setting
\[ \Delta_n^2 = P_{H_n+1} \Delta^2(M) P_{H_n+1}, \quad n = 0, 1, \ldots, \]
then
\[ \Delta_n^2 = P_{H_n+1} - \sum_{i=1}^d M_{z_i} P_{H_n} M_{z_i}. \]
Since
\[ \Delta^2(M) = P_{V_0} + \sum_{n=0}^\infty \Delta_n^2 \]
and each \( \Delta_n^2 \) is positive, by the assumption we have
\[ \text{Trace} \Delta^2(M) = \dim V_0 + \sum_{n=0}^\infty \text{Trace} \Delta_n^2 < \infty. \]

Just as shown by (2.1),
\[ \text{Trace} \Delta_n^2 = \frac{(n + 1) \dim V_{n+1} - (n + d) \dim V_n}{n + 1} \geq 0. \]

Since the graded submodule \( M \) is necessarily generated by finitely many homogeneous polynomials, this implies that there exists a large positive integer \( N_0 \) such that \( V_{n+1} = z_1 V_n + \cdots + z_d V_n \) if \( n \geq N_0 \). Since
\[ \sum_{n=N_0}^\infty \text{Trace} \Delta_n^2 = \sum_{n=N_0}^\infty \frac{(n + 1) \dim V_{n+1} - (n + d) \dim V_n}{n + 1} < \infty, \]
and the numerators of terms in this series are nonnegative integers, using the fact \( \sum_{n=N_0}^\infty \frac{1}{n + 1} = \infty \) shows that there is an integer \( N_1 \geq N_0 \) such that
\[ (N_1 + 1) \dim V_{N_1+1} = (N_1 + d) \dim V_{N_1}. \]

Applying Proposition 2.3 gives \( V_{N_1} = H_{N_1} \). It follows from this fact that \( M \) is finite codimensional. This completes the proof.

We will now prove the following result concerning two-dimensional case.

**Theorem 2.4.** Let \( M \) be a graded submodule of \( H_2^2 \). Then the submodule \( M \) belongs to every Schatten class \( \mathcal{L}^p \) for \( p > 2 \).

**Remark.** By Theorem 2.4 and Corollary 2.2, we have
\[ \mathcal{G}^2_{-\infty} = \mathcal{G}^2_0 = \mathcal{G}^2_2 = \mathcal{G}^2_p = \mathcal{G}^2_\infty, \quad p > 2. \]
We point out that Theorem 2.4 depends on the dimension $d = 2$, which will be shown by the next example.

**Example 3.** Considering the submodule $M = [z_1]$ of $H^2_d$ generated by $z_1$, then it is easy to verify that the reproducing kernel of $M^\perp$ is given by

\[ K_M^{\perp} = P_{M^\perp}K_\lambda = \frac{1}{1 - \lambda_1 z_1 - \lambda_2 z_2 - \lambda_3 z_3}, \]

and hence the reproducing kernel of $M$ is

\[ K_M = P_M K_\lambda = \frac{1}{1 - \lambda_1 z_1 - \lambda_2 z_2 - \lambda_3 z_3}. \]

By (1.4), we have

\[ \Delta^2(M) = \sum_{n \geq 0} \frac{n!}{n!(n-j)!} \lambda_1^{j} \lambda_2^{n-j} z_1^j z_2^{n-j} \otimes z_1 z_2 z_3. \]

Set $e_{jn} = z_1 z_2^{n-j} / \| z_1 z_2^{n-j} \|$ Then \{e_{jn}\} are mutually orthogonal, and it follows that

\[ \Delta(M) = \sum_{n \geq 0} \frac{n!}{n!(n-j)!} z_1 z_2^{n-j} \otimes z_1 z_2 z_3. \]

Moreover, we consider the submodule $N = [z_2, z_3]$ of $H^2_d$ generated by $z_2, z_3$. Then, as shown by Example 2, $\Delta(N)$ is in Schatten class $L^p$ if and only if $p > 2$.

To prove Theorem 2.4 we need the next lemma which was proved in [Yang].

**Lemma 2.5.** Let $I$ be an ideal of $C[z_1, z_2]$. If the greatest common divisor $\text{GCD}(I)$ of $I$ equals 1, then $I$ is a finite codimensional ideal.

**Proof of Theorem 2.4.** Let $M$ be a graded submodule of $H^2_d$. Then $M$ is necessarily generated by some homogeneous ideal $I$ of $C[z_1, z_2]$, that is, $M$ is equal to the closure of $I$. 

Guo, Defect operators for submodules of $H^2_d$
Let \( q \) be the greatest common divisor of \( I \). Then \( I \) can be written as \( I = qL \), where both \( q \) and \( L \) are homogeneous (cf. \([ZS]\)). Notice that the ideal \( L \) has the greatest common divisor 1. Therefore, by Lemma 2.5, \( L \) is a finite codimensional ideal of \( \mathbb{C}[z_1, z_2] \). This implies that there is a finite dimensional space \( F \) of polynomials such that

\[
\mathbb{C}[z_1, z_2] = L + F,
\]

and hence

\[
q\mathbb{C}[z_1, z_2] = qL + qF.
\]

So, the submodule \([q]\) generated by \( q \) is equal to \( M + qF \). Since \( qF \) is finite dimensional, this means that for \( p > 0 \), \( \Delta(M) \in \mathcal{L}^p \) if and only if \( \Delta([q]) \in \mathcal{L}^p \). Below we will verify that for the homogeneous polynomials \( q \), \( \Delta([q]) \in \mathcal{L}^p \) for each \( p > 2 \). Let \( l = \deg q \). Then \([q]\) can be decomposed as

\[
[q] = qH_0 \oplus qH_1 \oplus qH_2 \oplus \cdots.
\]

Write \( M_n \) for \( qH_n \). Then

\[
\Delta^2([q]) = P_{M_0} + \sum_{n=0}^{\infty} \left[ P_{M_{n+1}} - \sum_{i=1}^{2} M_{z_i} P_{M_n} M_{z_i}^* \right] .
\]

Putting

\[
\Delta^2_n = P_{H_{n+l+1}} \Delta^2([q]) P_{H_{n+l+1}},
\]

then we have

\[
\Delta^2_n = P_{M_{n+1}} - \sum_{i=1}^{2} M_{z_i} P_{M_n} M_{z_i}^* .
\]

Since \( \Delta^2_n \geq 0 \) and range \( \Delta^2_n \subseteq M_{n+1} \), there are normalized eigenvectors \( \{r_{n+1}^{(j)}\} (\in M_{n+1}) \) of \( \Delta^2_n \) with the corresponding eigenvalues \( \mu_{n+1,j} \) such that \( \{r_{n+1}^{(j)}\} \) are mutually orthogonal, and

\[
\Delta^2_n = \sum_j \mu_{n+1,j} r_{n+1}^{(j)} \otimes r_{n+1}^{(j)} .
\]

By (2.1), we have

\[
\text{Trace} \, \Delta^2_n = \dim M_{n+1} - \frac{n + l + 2}{n + l + 1} \dim M_n
\]

\[
= \dim qH_{n+1} - \frac{n + l + 2}{n + l + 1} \dim qH_n
\]

\[
= n + 2 - \frac{n + l + 2}{n + l + 1} (n + 1)
\]

\[
= \frac{l}{n + l + 1} .
\]
Also notice that
\[
\text{Trace } \Delta^2_n = \sum_j \mu_{n+1,j}
\]
and hence
\[
\sum_j \frac{n+l+1}{l} \mu_{n+1,j} = 1.
\]
This implies that
\[
\sum_j \left( \frac{n+l+1}{l} \right)^p \mu_{n+1,j}^p \leq 1
\]
whenever \( p \geq 1 \). Thus, if \( p \geq 1 \), then we have
\[
\sum_j \mu_{n+1,j}^p \leq \left( \frac{l}{n+l+1} \right)^p.
\]
Note that
\[
\Delta([q]) = P_{M_0} + \sum_{n \geq 0} \sum_j \sqrt{\mu_{n+1,j} r_{n+1}^{(j)} \otimes r_{n+1}^{(j)}}.
\]
It follows that for \( p > 2 \) we have
\[
\Delta^p([q]) = P_{M_0} + \sum_{n \geq 0} \sum_j \mu_{n+1,j}^{p/2} r_{n+1}^{(j)} \otimes r_{n+1}^{(j)},
\]
and in this case,
\[
\text{Trace } \Delta^p([q]) = 1 + \sum_{n \geq 0} \sum_j \mu_{n+1,j}^{p/2} \leq 1 + \sum_{n \geq 0} \left( \frac{l}{n+l+1} \right)^{p/2} < \infty.
\]
This shows that \( \Delta([q]) \) belongs to Schatten class \( \mathcal{L}^p \) for each \( p > 2 \), and hence \( \Delta(M) \) belongs to Schatten class \( \mathcal{L}^p \) for each \( p > 2 \). The proof is complete.

3. On a problem of Arveson

Let \( H \) be a contractive Hilbert module with canonical operators \( T_1, \ldots, T_d \). Considering the completely positive map \( P \) on \( \mathcal{B}(H) \) defined by
\[
P(A) = T_1 A T_1^* + \cdots + T_d A T_d^*, \quad A \in \mathcal{B}(H),
\]
then we have
Thus

\[ T_\infty = \lim P^n(I) \]

exists as a limit in the strong operator topology and satisfies \( 0 \leq T_\infty \leq I \). The Hilbert module \( H \) is called pure if \( T_\infty = 0 \). Recall that in [Arv2], a rank one free Hilbert module is defined as \( H_d^2 \), and a rank \( r \) free Hilbert module as \( r \cdot H_d^2 \) \((1 \leq r \leq \infty)\), where \( r \cdot H_d^2 \) denotes the direct sum of \( r \) copies of \( H_d^2 \). By the dilation theorem [Arv1], [Arv2], each pure contractive Hilbert module \( H \) with rank \( r \) is unitarily equivalent to a quotient of the rank \( r \) free Hilbert module. This means that there exists a closed submodule \( N \) of \( r \cdot H_d^2 \) such that

\[ \text{(3.1)} \quad 0 \to N \to r \cdot H_d^2 \to H \to 0 \]

is a short exact sequence of Hilbert modules. Therefore, the module \( H_d^2 \) plays the role of universal object in the category of pure contractive Hilbert modules. It suggests we should focus attention on the module \( H_d^2 \) and its submodules. As we have seen, the module \( H_d^2 \) has rank 1, and hence, each finite codimensional submodule must also have finite rank, and the ranks of such finite codimensional submodules can be arbitrarily large. Furthermore, Arveson has shown that if a graded submodule has finite rank, then the submodule must have finite codimension in \( H_d^2 \) [Arv2]. Since a quotient of the rank \( r \) free Hilbert module is a Hilbert module of rank at most \( r \), this means that for a graded submodule \( M \) with infinite codimension, one cannot find a finite integer \( r \) such that the short sequence (3.1) of Hilbert modules is exact when \( M \) is placed at the end of the sequence. This result stands in rather stark contrast with the assertion of Hilbert’s basis theorem for finitely generated modules over Noetherian rings. Let us recall the corresponding result in commutative algebra. Assume \( R \) is a Noetherian ring (that is, \( R \) is a commutative ring with unit \( e \), and any ideal of \( R \) is finitely generated). Then \( F = R \) will be the rank one free module over \( R \) and a rank \( r \) free module \( r \cdot F \) is the direct sum of \( r \) copies of \( F \), where \( 1 \leq r < \infty \). Assume \( M \subset r \cdot F \) is a submodule. Then by Hilbert basis theorem [ZS], \( M \) admits a finite system of generators \( \{a_1, \ldots, a_n\} \). Considering the free module \( n \cdot F \), then there exists a unique surjective module map \( \tau : n \cdot F \to M \) which maps the canonical basis vectors \( e_i \) to \( a_i \) \((i = 1, \ldots, n)\). Setting \( N = \ker \tau \), then \( N \) is a submodule of \( n \cdot F \) such that

\[ \text{(3.2)} \quad 0 \to N \to n \cdot F \to M \to 0 \]

is a short exact sequence of modules, that is, each submodule of a finite rank free module is isomorphic to a quotient of a finite rank free module. The above comparison suggests Hilbert modules may behave very different from algebraic modules.

Returning to submodules of \( H_d^2 \), a natural problem was asked by Arveson in [Arv2], [Arv4].

**Problem.** In dimension \( d \geq 2 \), must the rank of each nonzero submodule of \( H_d^2 \), which has infinite codimension in \( H_d^2 \), be infinite?

The author noticed Fang’s paper [Fa1] posted on arXiv. Fang attacked the above problem in this paper, and there are some clever ideas in the paper. However, there is a
nontrivial gap in page 10 of the paper. In this section we will combine function theory in the unit ball, the theory of ideals of polynomials and some techniques in Fang’s paper to give an affirmative answer to the above Problem.

**Theorem 3.1.** In dimension \( d \geq 2 \), if a nonzero submodule \( M \) of \( H^2_d \) is of finite rank, then the submodule has finite codimension in \( H^2_d \).

**Remark.** Theorem 3.1 applies here to show that each infinite codimension submodule is never unitarily equivalent to a quotient of a finite rank free Hilbert module. Equivalently, if the submodule \( M \) is of infinite codimension, then one cannot find a finite integer \( r \) such that the short sequence \((3.1)\) of Hilbert modules is exact when \( M \) is placed at the end of the sequence.

For this theorem we need some preliminaries. First we decompose \( M_{z_j} \) \((j = 1, \ldots, d)\) with respect to \( M \) and \( M^\perp \) as:

\[
M_{z_j} = \begin{pmatrix} R_{z_j} & J_{z_j} \\ 0 & S_{z_j} \end{pmatrix}.
\]

As noticed by Fang [Fa1], Jumping operators \( J_{z_j} : M^\perp \rightarrow M \ (1 \leq j \leq d) \) play an important role in the study of defect operators.

Observe that

\[
1 \otimes 1 = I - \sum_{j=1}^{d} M_{z_j} M_{z_j}^* = \begin{pmatrix} I_M - \sum_{j=1}^{d} R_{z_j} R_{z_j}^* - \sum_{j=1}^{d} J_{z_j} J_{z_j}^* & * \\ * & * \end{pmatrix},
\]

and

\[
\Delta^2(M) = I_M - \sum_{j=1}^{d} R_{z_j} R_{z_j}^*.
\]

This implies that \( \Delta(M) \) has finite rank if and only if the operator \( \sum_{j=1}^{d} J_{z_j} J_{z_j}^* \) is of finite rank, which in turn is equivalent to

\[
\sum_{j=1}^{d} \text{rank} J_{z_j} < \infty.
\]

Since

\[
J_{z_j} = P_M M_{z_j} P_M = P_M M_{z_j} - M_{z_j} P_M = [P_M, M_{z_j}], \quad j = 1, \ldots, d,
\]

this means that the defect operator \( \Delta(M) \) has finite rank if and only if commutators \( [P_M, M_{z_j}] \) have finite ranks for \( j = 1, \ldots, d \).

Let \( \mathcal{H}_j \) consist of analytic functions on \( \mathbb{B}_d \) involving only constants and variables \( z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d \). Then \( M \cap \mathcal{H}_j \) is closed and
As done by Fang [Fa1], decompose $M$ as

$$M = z_j M \oplus M \cap \mathcal{H}_j \oplus \Lambda_j.$$  

Since

$$\Lambda_j \subseteq P_{M \oplus z_j M} (M \cap \mathcal{H}_j) = P_{M \oplus z_j M} M^\perp + z_j H_d^2$$

$$= P_{M \oplus z_j M} M^\perp + z_j M + z_j M^\perp \subseteq P_{M \oplus z_j M} z_j M^\perp$$

$$= P_{M \oplus z_j M} z_j M^\perp \subseteq P_{M \oplus z_j M} \text{range } J_z,$$

it follows that if the defect operator $\Delta(M)$ has finite rank, then each $\Lambda_j$ is finite dimensional.

Furthermore, one can naturally identify $\mathcal{H}_j \cap H_d^2$ with the module $H_d^2$ in the variables $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d$. Therefore, if $M$ is a submodule of $H_d^2$, then $M \cap \mathcal{H}_j$ is a submodule of $H_d^2$.

**Lemma 3.2.** Assume $M$ is a submodule of $H_d^2$, and $\Delta(M)$ has finite rank. Then $\Delta(M \cap \mathcal{H}_j)$ has finite rank for $j = 1, \ldots, d$, where $M \cap \mathcal{H}_j$ is regarded as a submodule of $H_d^2$.

**Proof.** We will prove the lemma in the case $j = d$, and then the same reasoning applies to the general case. Let $N = M \cap \mathcal{H}_d$, and $N^\perp$ denote the orthogonal complement of $N$ in $H_d^2$. Considering the decomposition of $M$

$$M = z_d M \oplus N \oplus \Lambda_d,$$

then $N^\perp$, regarded as subspace of $H_d^2$, is closed, and is contained in $M^\perp + \Lambda_d$ (noticing that $\Lambda_d$ is finite dimensional). For $1 \leq i \leq d - 1$, considering the Jumping operator $J'_{z_i} = P_N M_{z_i} P_{N^\perp}$ on the space $H_d^2$, we have

$$J'_{z_i} \xi = P_N M_{z_i} P_{N^\perp} \xi = P_N P_M M_{z_i} P_{M^\perp + \Lambda_d} P_{N^\perp} \xi$$

for $\xi \in H_d^2$. From the preceding discussion we know that $J_{z_i} = P_M M_{z_i} P_{M^\perp}$ is of finite rank, and $\Lambda_d$ has finite dimension. This implies that $J'_{z_i}$ has finite rank for $i = 1, \ldots, d - 1$, and hence $\Delta(M \cap \mathcal{H}_d)$ has finite rank.

We also need the following lemma.

**Lemma 3.3.** In dimension $d \geq 2$, we have

$$\dim M / z_1 M + \cdots + z_j M + \cdots + z_d M = \infty,$$

for $j = 1, \ldots, d$, where the hat means that the term is omitted.
Remark. In Fang’s paper [Fa1], Fang proved \( \dim M / z_jM = \infty \) for each \( j \). The present proof is not distinct from that of Fang.

Proof. For a nonzero analytic function \( f \) on a neighborhood of the origin, let 
\[
 f(z_1, \ldots, z_d) = \sum_{\alpha} f_\alpha(z_d) z_1^{\alpha_1} \cdots z_d^{\alpha_{d-1}}
\]
be the expansion of \( f \) in the variables \( z_1, \ldots, z_{d-1} \), where 
\[
 z_1^{\alpha_1} \cdots z_d^{\alpha_{d-1}} = z_1^{\alpha_1} \cdots z_d^{\alpha_{d-1}}.
\]
We define the order of \( f \) in the variables \( z_1, \ldots, z_{d-1} \) as follows:
\[
 \text{order}(f) = \min \{ \alpha_1 + \cdots + \alpha_{d-1} : f_\alpha(z_d) \neq 0 \}.
\]
For the submodule \( M \), we define
\[
 \text{order}(M) = \min \{ \text{order}(f) : f \in M, f \neq 0 \}.
\]
Below, we will prove
\[
 \dim M / z_1M + \cdots + z_{d-1}M = \infty,
\]
and use the same argument to finish the general case. Set \( l = \text{order}(M) \), then it is easily verified \( \text{order}(z_1M + \cdots + z_{d-1}M) \geq l + 1 \). Assume \( h \in M \) achieves the value \( l \). We claim that the canonical map
\[
 \tau : C[z_1, \ldots, z_d]/(z_1C[z_1, \ldots, z_d] + \cdots + z_{d-1}C[z_1, \ldots, z_d])h \to M / z_1M + \cdots + z_{d-1}M
\]
is injective, where \( \tau \) maps \( \tilde{ph} \) to \( \tilde{ph} \). Indeed, for a polynomial \( p \) if \( ph \in z_1M + \cdots + z_{d-1}M \), then
\[
 p(0, \ldots, 0, z_d) h \in z_1M + \cdots + z_{d-1}M.
\]
If \( p(0, \ldots, 0, z_d) \neq 0 \), we would have
\[
 \text{order}(h) = \text{order}(p(0, \ldots, 0, z_d)h) \geq l + 1.
\]
This contradiction shows \( p(0, \ldots, 0, z_d) = 0 \), and hence
\[
 ph \in (z_1C[z_1, \ldots, z_d] + \cdots + z_{d-1}C[z_1, \ldots, z_d])h.
\]
The claim is proved. Combining the claim and the fact that
\[
 \dim C[z_1, \ldots, z_d]/(z_1C[z_1, \ldots, z_d] + \cdots + z_{d-1}C[z_1, \ldots, z_d]) = \infty,
\]
the desired conclusion follows.

We also will require some material on automorphisms of the unit ball \( \mathbb{B}_d \). For a more detailed exposition, we refer the reader to [Ru], Chapter 2. For each \( \lambda \in \mathbb{B}_d \) one can canonically define an automorphism \( \phi_\lambda \) in \( \text{Aut}(\mathbb{B}_d) \), the group of all automorphisms (biholomorphic mappings) of \( \mathbb{B}_d \) such that
(1) \( \phi_{\lambda}(0) = \lambda, \quad \phi_{\lambda}(\lambda) = 0 \);

(2) \( \phi_{\lambda} \circ \phi_{\lambda}(z) = z \),

and each \( \phi \in \text{Aut}(\mathbb{B}_d) \) has the form \( \phi = u \phi_{\lambda} \), where \( u \) is a unitary operator on \( \mathbb{C}^d \), and \( \lambda = \phi^{-1}(0) \). In fact, one can explicitly write \( \phi_{\lambda} \) as follows: if \( \lambda \neq 0 \),

\[
\phi_{\lambda}(z) = \frac{-\sqrt{1 - |\lambda|^2} \lambda \overline{z} + (\sqrt{1 - |\lambda|^2} - 1) \langle \lambda, \lambda \rangle \lambda + |\lambda|^2 \lambda}{|\lambda|^2 (1 - \langle \lambda, \lambda \rangle)},
\]

and \( \phi_0(z) = -z \). For each \( \mu \in \mathbb{B}_d \), one can define a unitary operator on \( H^2_d \) by

\[
V_\mu f = f \circ \phi_{\mu} k_\mu,
\]

where \( k_\mu \) is the normalized reproducing kernel of \( H^2_d \), \( k_\mu = K_\mu / \| K_\mu \| \). To verify that \( V_\mu \) is a unitary operator, from [Ru], Theorem 2.2.2(iii), a direct verifying gives

\[
K_\lambda \circ \phi_{\mu} k_\mu = \| K_\mu \| (1 - \langle \mu, \phi_{\mu}(\lambda) \rangle) K_{\phi_{\mu}(\lambda)},
\]

and letting \( \lambda = \mu \) in (3.4) implies

\[
k_\mu \circ \phi_{\mu} k_\mu = 1.
\]

Using (3.4), it is easy to check that

\[
\langle V_\mu K_\lambda, V_{\mu'} K_{\lambda'} \rangle = \langle K_\lambda, K_{\lambda'} \rangle,
\]

and by (3.5), we have \( V^2_\mu = I \). This shows that \( V_\mu \) is a unitary operator (this fact also was noticed by Greene [Gr]).

For a submodule \( M \), writing \( M_\mu \) for \( V_\mu M \), then, clearly,

\[
M_\mu = M \circ \phi_u = \{ f \circ \phi_u : f \in M \},
\]

and \( M_\mu \) is a submodule. Furthermore, we have

\[
P_{M_\mu} = V_\mu P_M V_\mu.
\]

**Lemma 3.4.** Let \( M \) be a submodule of \( H^2_d \). Then for every \( \mu \in \mathbb{B}_d \), we have

\[
\text{rank } \Delta(M) = \text{rank } \Delta(M_\mu).
\]

**Proof.** First, we claim that each function in \( \Delta(M) \cap H^2_d \) is a multiplier of \( H^2_d \). In fact, since the submodule \( M \) is contractive, by the Dilation Theorem 1.4 in [Arv2] there exists a unique bounded module homomorphism

\[
L : H^2_d \otimes \overline{\Delta(M)H^2_d} \to M
\]

satisfying
Given an $h \in H^2_d$, and setting $\phi = \Delta^2(M)h$, then

$$\phi = L(1 \otimes \Delta(M)h).$$

For any $f \in H^2_d$, pick a sequence $\{p_n\}$ of polynomials such that $p_n$ converges to $f$ in the norm of $H^2_d$. This implies that

$$p_n\phi = L(p_n \otimes \Delta(M)h) \to L(f \otimes \Delta(M)h)$$

as $n \to \infty$. Since for any $z \in \mathbb{B}_d$,

$$\phi(z)p_n(z) \to \phi(z)f(z)$$

as $n \to \infty$, this means that $\phi f = L(f \otimes \Delta(M)h) \in M$. Then a simple application of the closed graph theorem shows that $M_\phi$ is bounded multiplication operator, and hence the claim follows. Below, we will show:

$$\text{rank} \Delta(M_\mu) \leq \text{rank} \Delta(M).$$

We may assume $\text{rank} \Delta(M) < \infty$, and hence

$$\text{rank} \Delta^2(M) = \text{rank} \Delta(M) < \infty.$$

Pick $\phi_1, \ldots, \phi_l$ in $\Delta^2(M)H^2_d$ satisfying

$$\Delta^2(M) = \sum_{i=1}^l \phi_i \otimes \phi_i,$$

where $l = \text{rank} \Delta(M)$. Combining the claim and (1.4), we see

$$P_M K_\lambda = \left[ \sum_{i=1}^l M_{\phi_i} M_{\phi_i}^* \right] K_\lambda, \quad \lambda \in \mathbb{B}_d,$$

and thus,

$$P_M = \sum_{i=1}^l M_{\phi_i} M_{\phi_i}^*.$$

This yields

$$P_{M_\mu} = V_\mu \left[ \sum_{i=1}^l M_{\phi_i} M_{\phi_i}^* \right] V_\mu = \sum_{i=1}^l M_{\phi_i \circ \phi_\mu} M_{\phi_i \circ \phi_\mu}^*.$$

Using the argument following (1.5) gives that
This shows
\[
\operatorname{rank} \Delta(M_\mu) = \operatorname{rank} \Delta^2(M_\mu) \leq \operatorname{rank} \Delta(M).
\]
Since \((M_\mu)_\mu = M\), the same argument shows
\[
\operatorname{rank} \Delta(M) \leq \operatorname{rank} \Delta(M_\mu).
\]
We thus reach at the equality
\[
\operatorname{rank} \Delta(M) = \operatorname{rank} \Delta(M_\mu).
\]

**Proof of Theorem 3.1.** We will prove the theorem for \(d = 2\), and then use the induction argument to finish the general case. Noticing that
\[
M \cap \mathcal{H}_1 = M \cap H^2_{z_2}, \quad M \cap \mathcal{H}_2 = M \cap H^2_{z_1},
\]
where \(H^2_{z_i}\) denote the classical Hardy space on the unit disk in variable \(z_i\) for \(i = 1, 2\), we can decompose \(M\) as:
\[
M = \overline{z_2 M} \oplus M \cap H^2_{z_2} \oplus \Lambda_1, \quad M = \overline{z_1 M} \oplus M \cap H^2_{z_2} \oplus \Lambda_2.
\]
Then by the preceding discussion we have \(\dim \Lambda_i < \infty\) for \(i = 1, 2\). Applying Lemma 3.3 yields \(M \cap H^2_{z_i} \neq 0\) for \(i = 1, 2\). Pick nonzero \(f \in M \cap H^2_{z_i}\) and \(g \in M \cap H^2_{z_2}\), and let \(0 < \gamma < 1/2\) be such that
\[
f(\gamma) \neq 0, \quad g(\gamma) \neq 0.
\]
Set \(M_\gamma = M \circ \phi(\gamma, 0)\), where \(\phi(\gamma, 0) \in \operatorname{Aut}(\mathbb{B}_2)\), and by (3.3),
\[
\phi(\gamma, 0)(z) = \left( \frac{\gamma - z_1}{1 - \gamma z_1}, \frac{-\sqrt{1 - \gamma^2 z_2}}{1 - \gamma z_1} \right).
\]
Then by Lemma 3.4, we have
\[
\operatorname{rank} \Delta(M_\gamma) = \operatorname{rank} \Delta(M) < \infty,
\]
and hence \(M_\gamma\) can be decomposed as:
\[
M_\gamma = \overline{z_1 M_\gamma} \oplus M_\gamma \cap H^2_{z_2} \oplus L_1, \quad M_\gamma = \overline{z_2 M_\gamma} \oplus M_\gamma \cap H^2_{z_1} \oplus L_2
\]
with \(\dim L_i < \infty\) for \(i = 1, 2\). Putting \(f_\gamma(z_1) = f \circ \phi_\gamma(z_1)\), where \(\phi_\gamma(z_1) = \frac{\gamma - z_1}{1 - \gamma z_1}\), then \(f_\gamma(z_1) \in M_\gamma \cap H^2_{z_1}\), and \(f_\gamma(0) = f(\gamma) \neq 0\). Let
\[
\mathcal{L} = \{ p(z_2) f_\gamma(z_1) : p(z_2) \in C[z_2] \} \subseteq M_\gamma.
\]
Since the projection $P_{L_1} \mid \varphi : \mathcal{L} \to L_1$ has finite dimensional range, noticing the first decomposition in (3.7) shows that there exists a nonzero polynomial $p(z_2)$ such that

$$p(z_2)f_j(z_1) \in z_1 \overline{M_f} \oplus M_f \cap H^2_{z_2}.$$ 

Hence, there are $\psi(z_2) \in M_f \cap H^2_{z_2}$, and $\phi(z_1, z_2) \in z_1 \overline{M_f}$ such that

$$p(z_2)f_j(z_1) = \psi(z_2) + \phi(z_1, z_2).$$ 

Letting $z_1 = 0$ gives that $\psi(z_2) = f(\gamma)p(z_2) \in M_f \cap H^2_{z_2}$, and therefore, $p(z_2) \in M_f \cap H^2_{z_2}$. Since $M_f \cap H^2_{z_2}$ is a closed invariant subspace of the classical Hardy space $H^2_{z_2}$ on the unit disk, and it contains a nonzero polynomial, this insures that $M_f \cap H^2_{z_2}$ is finite codimensional in $H^2_{z_2}$, and it follows that there is a nonzero polynomial $p_1(z_2) \in M_f \cap H^2_{z_2}$ satisfying $Z(p_1) \subset \mathbb{D}$. Moreover, by (3.6), we have

$$p_1 \circ \phi_{(\gamma, 0)}(z) = p_1 \left( -\frac{\sqrt{1 - \gamma^2 z_2}}{1 - \gamma z_1} \right) \in M.$$ 

The same argument shows that there exists a nonzero polynomial $p_2(z_1)$ with $Z(p_2) \subset \mathbb{D}$ satisfying

$$p_2(z_1) \in M^\gamma \cap H^2_{z_1},$$

where $M^\gamma = M \circ \phi_{(0, \gamma)}$, and hence

$$p_2 \circ \phi_{(0, \gamma)}(z) = p_2 \left( -\frac{\sqrt{1 - \gamma^2 z_1}}{1 - \gamma z_2} \right) \in M.$$ 

Decomposing polynomials $p_1$ and $p_2$ as:

$$p_1(z) = \prod_{i=1}^s (z - a_i), \quad p_2(z) = \prod_{j=1}^t (z - b_j),$$

then we have

$$p_1 \left( -\frac{\sqrt{1 - \gamma^2 z_2}}{1 - \gamma z_1} \right) = \frac{(-1)^s}{(1 - \gamma z_1)^s} \prod_{i=1}^s (\sqrt{1 - \gamma^2 z_2} - a_i \gamma z_1 + a_i),$$

$$p_2 \left( -\frac{\sqrt{1 - \gamma^2 z_1}}{1 - \gamma z_2} \right) = \frac{(-1)^t}{(1 - \gamma z_2)^t} \prod_{j=1}^t (\sqrt{1 - \gamma^2 z_1} - b_j \gamma z_2 + b_j).$$

Therefore,

$$q_1 = \prod_{i=1}^s (\sqrt{1 - \gamma^2 z_2} - a_i \gamma z_1 + a_i) \in M, \quad q_2 = \prod_{j=1}^t (\sqrt{1 - \gamma^2 z_1} - b_j \gamma z_2 + b_j) \in M.$$
We claim that \( q_1 \) and \( q_2 \) are co-prime, that is, \( \gcd(q_1, q_2) = 1 \). In fact, if \( \gcd(q_1, q_2) \) is not constant, then there exist some \( a_i \) and \( b_j \) and a constant \( c \) such that
\[
\sqrt{1 - \gamma^2 z_2} - a_i \gamma z_1 + a_i = c(\sqrt{1 - \gamma^2 z_1} - b_j \gamma z_2 + b_j).
\]

From this equation we have
\[
a_i = b_j = -\sqrt{\frac{1}{\gamma^2} - 1}, \quad c = 1.
\]

Since \( 0 < \gamma < 1/2 \), we see \( |a_i| = |b_j| > 1 \). This is contrary to the fact \( Z(p_1) \cup Z(p_2) \subseteq \mathbb{D} \). Consequently, \( \gcd(q_1, q_2) = 1 \). By Lemma 2.5 we see that the ideal \( q_1 C[z_1, z_2] + q_2 C[z_1, z_2] \) is finite codimensional in \( C[z_1, z_2] \). This says that there is a finite dimensional space \( F \) of polynomials such that
\[
C[z_1, z_2] = q_1 C[z_1, z_2] + q_2 C[z_1, z_2] + F,
\]
and hence
\[
H^2_2 = \overline{C[z_1, z_2]} = [q_1, q_2] + F,
\]
where \([q_1, q_2]\) denotes the submodule generated by \( q_1 \) and \( q_2 \), that is, \([q_1, q_2]\) is the closure of \( q_1 C[z_1, z_2] + q_2 C[z_1, z_2] \) in the norm of \( H^2_2 \). This means that the submodule \([q_1, q_2]\) is finite codimensional in \( H^2_2 \). By the inclusion \([q_1, q_2] \subseteq M\), the submodule \( M \) is finite codimensional, completing the proof in the case \( d = 2 \).

Now using induction, assume our proof is finished for the \( d - 1 \) dimensional case. For \( j = 1, \ldots, d \), recall that \( M \cap \mathcal{H}_j \) is a submodule of \( H^2_{d-1} \) in the variables \( z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d \), and by Lemma 3.3, \( M \cap \mathcal{H}_j \neq 0 \). Applying Lemma 3.2, rank \( \Delta(M \cap \mathcal{H}_j) < \infty \) for \( j = 1, \ldots, d \), where \( M \cap \mathcal{H}_j \) is regarded as a submodule of \( H^2_{d-1} \). By the assumption \( M \cap \mathcal{H}_j \) is finite codimensional in \( H^2_{d-1} \) in the variables \( z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d \). In particular, this implies that in the case \( M \cap \mathcal{H}_d \), there exist nonzero polynomials \( p_1(z_1) \in M, \ldots, p_{d-1}(z_{d-1}) \in M \), and in the case \( M \cap \mathcal{H}_1 \), there exist nonzero polynomials \( q_2(z_2) \in M, \ldots, q_d(z_d) \in M \). Set
\[
I = p_1(z_1) C[z_1, \ldots, z_d] + q_2(z_2) C[z_1, \ldots, z_d] + \cdots + q_d(z_d) C[z_1, \ldots, z_d].
\]

Then the number \( |Z(I)| \) of zeros of the ideal \( I \) satisfies
\[
|Z(I)| \leq \deg p_1 \deg q_2 \cdots \deg q_d < \infty.
\]

Since \( I \) has finitely many zeros in \( C^d \), this implies that the ideal \( I \) is finite codimensional in \( C[z_1, \ldots, z_d] \), and hence the submodule \([I]\) generated by \( I \) is finite codimensional. Because \([I] \subseteq M\), this insures that \( M \) is a finite codimensional submodule, completing the proof of Theorem 3.1.
4. An analogous result for the reproducing kernel function space $\mathcal{H}_v^d$

In this section we will generalize Theorem 3.1 to the reproducing function space $\mathcal{H}_v^d$ ($v > 0$) on the unit ball of $\mathbb{C}^d$. Recall that by the reproducing kernel theory [Ar], for $v > 0$, the function

$$K^{(v)}_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^v}$$

defined on $\mathbb{B}_d \times \mathbb{B}_d$ is a reproducing kernel on $\mathbb{B}_d$. Therefore, it induces a reproducing kernel function space on $\mathbb{B}_d$, denoted by $\mathcal{H}_v^d$. Then, $\mathcal{H}_1^d$ is the symmetric Fock space $H^2_J$, $\mathcal{H}_v^d$ is the Hardy space $H^2(\mathbb{B}_d)$, and when $v > d$, $\mathcal{H}_v^d$ is the weighted Bergman space $L^2_v[(1 - |z|^2)^{v-d-1} \, dV]$ on the unit ball, where $dV$ is the volume measure, and in particular, $\mathcal{H}_d^d$ is the Bergman space $L^2_d(\mathbb{B}_d)$. The reader easily verifies that the $d$-tuple $\{M_{z_1}, \ldots, M_{z_d}\}$ acting on $\mathcal{H}_v^d$ is jointly bounded, and hence $\mathcal{H}_v^d$ admits a natural $C[z_1, \ldots, z_d]$-module structure. A routine verifying shows that the space $\mathcal{H}_v^d$ has a canonical orthonormal basis

$$\left\{ \left[ \frac{v(v+1) \cdots (v+|z|-1)}{\alpha!} \right]^{1/2} z^\alpha \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ run over all multi-indexes of nonnegative integers, and $\alpha! = \alpha_1! \cdots \alpha_d!$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. Using (4.1), the next equality follows:

$$\sum_{i=1}^d M_{z_i} M_{z_i}^* = \sum_{n=0}^{\infty} \frac{n + 1}{n + v} P_{H_{n+1}},$$

where $P_{H_n}$ is the orthogonal projection from $\mathcal{H}_v^d$ to the space of homogeneous polynomials with degree $n$. Therefore, the module $\mathcal{H}_v^d$ is contractive if and only if $v \geq 1$, and in this case, the module $\mathcal{H}_v^d$ is pure. To see this, let $A = \lim_{n \to \infty} P^n(I)$, and $k^{(v)}_\lambda = K^{(v)}_\lambda / \|K^{(v)}_\lambda\|$ be the normalized reproducing kernel. Then we have

$$\langle AK^{(v)}_\lambda, K^{(v)}_\lambda \rangle = \lim_{n \to \infty} \langle P^n(I)k^{(v)}_\lambda, k^{(v)}_\lambda \rangle = \lim_{n \to \infty} (|\lambda_1|^2 + \cdots + |\lambda_d|^2)^n = 0,$$

and hence $\langle AK^{(v)}_\lambda, K^{(v)}_\lambda \rangle = 0$ for any $\lambda \in \mathbb{B}_d$. By applying Lemma 1.2, the same proof as Proposition 1.1 shows $\langle AK^{(v)}_\lambda, K^{(v)}_\lambda \rangle = 0$, and hence $A = 0$. Moreover, from (4.2), we see that $\mathcal{H}_v^d$ is an infinite rank pure Hilbert module if $v > 1$.

Recall that for a submodule $M$ of $H^2_d$, the defect operator $\Delta(M)$ has finite rank if and only if

$$\sum_{j=1}^d \text{rank} J_{z_j} < \infty.$$  

Analyzing the proof of Theorem 3.1, from $\sum_{j=1}^d \text{rank} J_{z_j} < \infty$, the submodule has the required decomposition that in turn implies that the submodule has finite codimension.
Let $M$ be a submodule of $H^d_v$. Decompose $M_{z_j} (j = 1, \ldots, d)$ with respect to $M$ and $M^\perp$ as:

$$M_{z_j} = \begin{pmatrix} R_{z_j} & J_{z_j} \\ 0 & S_{z_j} \end{pmatrix}$$

as done in the case $H^2_d$. Then Jumping operators $J_{z_j} (1 \leq j \leq d)$ are given by

$$J_{z_j} = P_M M_{z_j} P_{M^\perp} = P_M M_{z_j} - M_{z_j} P_M = [P_M, M_{z_j}].$$

**Theorem 4.1.** Let $M$ be a nonzero submodule of $H^d_v$ ($d \geq 2$). Then

$$\sum_{j=1}^d \text{rank } J_{z_j} = \sum_{j=1}^d \text{rank}[P_M, M_{z_j}] < \infty$$

if and only if $M$ has finite codimension.

Concerning the proof of Theorem 4.1, although the proof comes from similar techniques used in the proof of Theorem 3.1, but several modifications are required. We will sketch the proof of this theorem.

For each $\mu \in \mathbb{B}_d$, we can define a unitary operator on $H^d_v$ by

$$V_{\mu} f = f \circ \phi_{\mu} k_{\mu}^{(v)}.$$  

To verify that $V_{\mu}$ is a unitary operator, from [Ru], Theorem 2.2.2(iii), a direct verifying shows

$$K_{\lambda}^{(v)} \circ \phi_{\mu} k_{\mu}^{(v)} = \|K_{\mu}^{(v)}\| K_{\beta}^{(v)}(z), \quad \text{here } \beta = \phi_{\mu}(\lambda),$$

and letting $\lambda = \mu$ in (4.4) implies

$$k_{\mu}^{(v)} \circ \phi_{\mu} k_{\mu}^{(v)} = 1.$$

Using (4.4), it is easy to check

$$\langle V_{\mu} K_{\lambda}^{(v)}, V_{\mu} K_{\lambda'}^{(v)} \rangle = \langle K_{\lambda}^{(v)}, K_{\lambda'}^{(v)} \rangle,$$

and by (4.5), we have $V_{\mu}^2 = I$. This insures that $V_{\mu}$ is a unitary operator.

For a submodule $M$ of $H^d_v$, writing $M_{\mu}$ for $V_{\mu} M$, then it is easy to verify that $M_{\mu}$ is a submodule, and

$$P_{M_{\mu}} = V_{\mu} P_M V_{\mu}.$$

**Lemma 4.2.** Let $M$ be a submodule of $H^d_v$, and $M_{\mu}$ as above. Then the following are equivalent:
(1) \( \sum_{k=1}^{d} \text{rank } J_{z_k} < \infty \);

(2) \( \sum_{k=1}^{d} \text{rank } J'_{z_k} < \infty \), where \( J'_{z_k} \) are Jumping operators of \( M_\mu \).

**Proof.** The reader easily checks that for each \( \mu \in \mathbb{B}_d \), \( g_\mu(z) = 1/(1 - \langle z, \mu \rangle) \) is a multiplier of \( \mathcal{H}_e^d \), that is, \( M_\mu \) acting on \( \mathcal{H}_e^d \) is bounded. To deduce (2) from (1) we claim that the operator \( P_M M_\mu P_M \to \) has finite rank. For the claim, let \( f_\mu(z) = 1 - \langle z, \mu \rangle \). Then we have

\[
0 = P_M M_\mu M_\mu P_M = P_M M_\mu (P_M + P_M^\perp) M_\mu P_M^\perp = P_M M_\mu P_M M_\mu P_M^\perp + P_M M_\mu P_M^\perp M_\mu P_M^\perp.
\]

This implies

\[
M_\mu P_M M_\mu P_M^\perp = -P_M M_\mu P_M^\perp M_\mu P_M^\perp,
\]

and hence

\[
P_M M_\mu P_M^\perp = -M_\mu P_M M_\mu P_M^\perp M_\mu P_M^\perp.
\]

By the fact rank \( P_M M_\mu P_M^\perp < \infty \), the operator \( P_M M_\mu P_M^\perp \) is of finite rank.

Let \( \phi_\mu(z) = (\phi^{(1)}_\mu(z), \ldots, \phi^{(d)}_\mu(z)) \). Then by (3.3) there are constants \( a_k^{(j)}(\mu), 1 \leq j \leq d \) and \( c_k(\mu) \) depending only on \( \mu \) such that

\[
\phi^{(k)}_\mu(z) = \frac{\sum_{j=1}^{d} a_k^{(j)}(\mu)z_j + c_k(\mu)}{1 - \langle z, \mu \rangle}
\]

for \( k = 1, \ldots, d \). Considering the Jumping operator \( J'_{z_k} \), we have

\[
J'_{z_k} = P_M M_{z_k} P_M^\perp = V_\mu P_M M_{z_k \circ \phi_\mu} P_M^\perp V_\mu = V_\mu P_M M_{\phi^{(k)}_\mu} P_M^\perp V_\mu.
\]

To show rank \( J'_{z_k} < \infty \) for \( 1 \leq k \leq d \), it suffices to prove

\[
\text{rank } P_M M_{\phi^{(k)}_\mu} P_M^\perp < \infty.
\]

Setting \( h_\mu(z) = \sum_{j=1}^{d} a_k^{(j)}(\mu)z_j + c_k(\mu) \), then we have

\[
P_M M_{\phi^{(k)}_\mu} P_M^\perp = P_M M_{h_\mu} M_\mu P_M^\perp = P_M M_{h_\mu} (P_M + P_M^\perp) M_\mu P_M^\perp = P_M M_{h_\mu} P_M^\perp M_\mu P_M^\perp + P_M M_{h_\mu} P_M^\perp M_\mu P_M^\perp.
\]
From the claim and rank $P_M M_{h_k} P_M < \infty$, the operator $P_M M_{\phi_{\mu_k}} P_M$ has finite rank for $k = 1, \ldots, d$. This shows that (1) implies (2). Since $(M_{\mu})_{\mu} = M$, the same argument shows that (2) implies (1).

We also need the following lemma.

**Lemma 4.3.** Let $M$ be a submodule of $\mathcal{H}_v^1$ on the unit disk $\mathbb{D}$. Then $M$ has finite codimension if and only if there is a polynomial $p(z)$ with $Z(p) \subset \mathbb{D}$ such that

$$M = p\mathcal{H}_v^1,$$

and in this case, the codimension of $M$ equals $\deg p$.

**Proof.** From the definition of analytic Hilbert modules (see [CG], p. 23), it is easy to verify that $\mathcal{H}_v^1$ is an analytic Hilbert module on the unit disk. Using [DPSY], Corollary 2.8, or [CG], Theorem 2.2.3, $M$ is a finite codimensional submodule of $\mathcal{H}_v^1$ if and only if there is a polynomial $p(z)$ with $Z(p) \subset \mathbb{D}$ such that $M = p\mathcal{H}_v^1$ and $\dim \mathcal{H}_v^1 / M = \deg p$. From the inclusion $Z(p) \subset \mathbb{D}$, the reader easily checks that

$$M = p\mathcal{H}_v^1.$$

As done in the case $H_2^2$, let $\mathcal{H}_j$ consist of analytic functions on $\mathbb{B}_d$ involving only constants and variables $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d$. Then $\mathcal{H}_v^d \cap \mathcal{H}_j$ is a closed subspace of $\mathcal{H}_v^d$, and it can be canonically identified with $\mathcal{H}_v^{d-1}$. For a submodule $M$ of $\mathcal{H}_v^d$, then $M \cap \mathcal{H}_j$ is a submodule of $\mathcal{H}_v^{d-1}$. Following the proof of Lemma 3.2, we have

**Lemma 4.4.** Let $M$ be a submodule of $\mathcal{H}_v^d$, and Jumping operators of $M$ satisfy (4.3). Then Jumping operators of $M \cap \mathcal{H}_j$ also satisfy (4.3) for $1 \leq j \leq d$, where $M \cap \mathcal{H}_j$ is regarded as a submodule of $\mathcal{H}_v^{d-1}$.

Also, the same proof as Lemma 3.3 shows

**Lemma 4.5.** In dimension $d \geq 2$, for a nonzero submodule $M$ of $\mathcal{H}_v^d$, we have

$$\dim M / z_1 M + \cdots + \bar{z}_j M + \cdots + z_d M = \infty,$$

for $j = 1, \ldots, d$, where the hat means that the term is omitted.

Now it is time to see the proof of Theorem 4.1. Combining Lemmas 4.2, 4.3, 4.4 and 4.5, and using induction argument as done in the proof of Theorem 3.1, Theorem 4.1 follows.

**Remark 1.** In Fang’s paper [Fa1], Fang proved Theorem 4.1 in the case of the Hardy module $H^2(\mathbb{D})$ over the polydisk. However, Fang’s proof relies heavily on Beurling’s theorem and the fact that $R_z = (R_{z_1}, \ldots, R_{z_d})$ are isometries. It is not difficult to see that using the same argument as in this section shows that Theorem 4.1 is valid for the Hardy modules and the Bergman modules over the polydisk, and even for the Hardy modules and the Bergman modules over bounded symmetric domains.
Remark 2. For every nontrivial submodule $M$ of $H^2(\mathbb{D})$, we have
\[ \text{rank}[P_M, M_2] = 1. \]

For a nontrivial submodule $M$ of the Bergman module $L^2_a(\mathbb{D})$, we do not know if\[ \text{rank}[P_M, M_2] < \infty \]
implies that $M$ has finite codimension.

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References

Guo, Defect operators for submodules of $H^2_d$


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