Essentially normal Hilbert modules and K-homology II: Quasi-homogeneous Hilbert modules over the two dimensional unit ball

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Abstract

In this paper, we mainly consider quasi-homogeneous submodules of $U$-invariant analytic Hilbert modules over the two dimensional unit ball. It is shown that every quasi-homogeneous submodule $M$ is essentially normal. This paper also shows that each quasi-homogeneous submodule of the Bergman module $L^2_a(B_2)$ is $p$-essentially normal for $p > 2$, and the same result also is valid for the Hardy module. The paper is associated with $K$-homology invariants arising from quasi-homogeneous quotient modules.

1 Preliminaries

The study of Hilbert modules is a natural approach to multivariable operator theory [DP]. Given a tuple of commuting operators $(T_1, \ldots, T_d)$ acting on a Hilbert space $H$, there is a natural way to make $H$ into a $C[z_1, \ldots, z_d]$-module as follows:

$$p \cdot h = p(T_1, \ldots, T_d)h, \quad \text{for } p \in C[z_1, \ldots, z_d], h \in H.$$ 

In the paper [GW], we studied essential normality and $p$-essential normality of homogeneous Hilbert modules over the unit ball $1$. This paper will continue this work. We will be interested in submodules of a $U$-invariant analytic Hilbert module $H$ over the unit ball $B_2 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$. That $H$ is $U$-invariant means that for each unitary map $U : \mathbb{C}^2 \to \mathbb{C}^2$, the operator $\Gamma_U : H \to H$ defined by $\Gamma_U f(z, w) = f(U(z, w))$ is unitary. Equivalently, the reproducing kernel of $H$ satisfies $K_{U(\alpha, \beta)}(U(z, w)) = K(\alpha, \beta)(z, w)$ for all $(\alpha, \beta), (z, w) \in B_2$. By [GHX], there exists a unique power series

\footnotesize
\begin{itemize}
  \item [0] 2000 AMS Subject Classification: 47A13; 47A20; 46H25; 46C99.
  \item [1] Here we follow Arveson’s terminology [Ar3, Ar4, Ar5], while Douglas and Paulsen used the term “essentially reductive” [DP], and Douglas the term “$p$-essentially reductive” [Dou2, Dou3].
\end{itemize}

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\[ f(z) = \sum_n a_n z^n \] with nonnegative coefficients having radius of convergence \( R \geq 1 \) such that
\[ K_{(\alpha,\beta)}(z, w) = f(\bar{\alpha}z + \bar{\beta}w). \]

Moreover, the function space \( H \) is called an analytic Hilbert module over \( \mathbb{B}_2 \) if the following conditions are satisfied:

1. the polynomial ring \( \mathbb{C}[z, w] \) is dense in \( H \);
2. for each polynomial \( p \), the multiplication operator \( M_p \) acting on \( H \) is bounded;
3. the evaluation functional \( E_{(\alpha,\beta)} : f \to f(\alpha, \beta) \) is bounded if and only if \( (\alpha, \beta) \in \mathbb{B}_2 \).

Therefore, an analytic Hilbert module over \( \mathbb{B}_2 \) is a module over the polynomial ring \( \mathbb{C}[z, w] \). The above condition (3) is equivalent to the condition that
\[ \|E_{(\alpha,\beta)}\|^2 = \|K_{(\alpha,\beta)}\|^2 = \sum_n a_n (|\alpha|^2 + |\beta|^2)^n \to \infty \]
as \( (\alpha, \beta) \to \partial \mathbb{B}_2 \), that is,
\[ \sum_n a_n = \infty. \] (1.1)

In this case, it is easy to verify that if \( (\alpha, \beta) \to \partial \mathbb{B}_2 \), then \( k_{(\alpha,\beta)} \to 0 \), where \( k_{(\alpha,\beta)} = \frac{K_{(\alpha,\beta)}}{\|K_{(\alpha,\beta)}\|} \) is the normalized reproducing kernel.

By a submodule of \( H \), we mean that it is a closed invariant subspace of \( H \) for multiplications by polynomials. In what follows we mainly consider the quasi-homogeneous submodules.

Given a \( d \)-tuple \( (m_1, \ldots, m_d) \) of positive integers, a polynomial in \( d \)-variables
\[ p(z_1, \ldots, z_d) = \sum_{i_1, \ldots, i_d} a(i_1, \ldots, i_d) z_1^{i_1} \cdots z_d^{i_d} \]
is said to be quasi-homogeneous with weight \( (m_1, \ldots, m_d) \) if there exists an integer \( l \) such that
\[ \sum_{k=1}^d m_k i_k = l \]
for all \( a(i_1, \ldots, i_d) \neq 0 \), and \( l \) is called the degree of \( p \) with respect to the weight \( (m_1, \ldots, m_d) \). The study of quasi-homogeneous polynomials has
drawn much interest in algebraic geometry and topology. An interesting example is the quasi-homogeneous polynomial \( f : \mathbb{C}^d \to \mathbb{C}^d \) defined by

\[
f(z_1, \ldots, z_d) = z_1^{a_1} + \cdots + z_d^{a_d},
\]

where the integers \( a_1, \ldots, a_d \geq 2 \). In [M], Milnor showed that the zero variety \( \Sigma_a = \{ z \in \mathbb{C}^d : f(z) = 0 \} \cap \partial B_d \) is a \((2d - 3)\) dimensional manifold. It is well known that, for many suitable \( a_1, \ldots, a_d \) and \( d \neq 3 \), \( \Sigma_a \) is a topological sphere. Milnor [M] and Brieskorn [Bri] gave a necessary and sufficient condition for \( \Sigma_a \) to be a topological sphere or an exotic sphere for \( d \neq 3 \). In fact, each exotic sphere which embeds in codimension 2 can be obtained in this way. In dimension \( d = 3 \), it is still an attractive field in low dimensional topology. R. Douglas suggested to study connections between the theory of exotic spheres and geometric analysis of Hilbert modules during a visit at Fudan University.

In dimension \( d = 2 \), if \( p \) is quasi-homogeneous with the weight \((m, n)\) and degree \( l \), it is easy to see \( p \) is quasi-homogeneous with respect to weight \((m, n)\) and \( \deg(p) = l/k \). Thus, in what follows we always assume the quasi-homogeneous weight \((m, n)\) satisfies \( GCD(m, n) = 1 \). A simple example of quasi-homogeneous polynomials is \( p = z^n - \alpha w^m \) with \( \deg(p) = mn \). In fact, by [BM], any quasi-homogeneous polynomial \( p \) with the weight \((m, n)\) can be uniquely decomposed as

\[
p(z, w) = \alpha z^u w^v \prod (z^n - \alpha_i w^m)^{s_i}
\]

with \( \deg(p) = mu + nv + mn \sum_i s_i \).

For a \( \mathcal{U} \)-invariant analytic Hilbert module \( H \) over \( \mathbb{B}_2 \) defined by the power series \( f(z) = \sum_n a_n z^n \), by [GHX], \( H \) has a canonical orthonormal basis

\[
\{ [a_{i+j}(i+j)!]^{1/2} z^i w^j : i, j \in \mathbb{Z}_+ \}.
\]

It follows that the whole space \( H \) admits an orthogonal quasi-homogeneous decomposition with respect to weight \((m, n)\) as follows:

\[
H = \oplus \mathbb{H}_l,
\]

(1.2)

where \( \mathbb{H}_l = \{ \sum_{i,j} a_{i,j} z^i w^j : mi + nj = l \} \) is all quasi-homogeneous polynomials with degree \( l \).

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Let $M$ be a submodule of an analytic Hilbert module $H$. Then $H \ominus M$ admits a quotient module structure induced by $\{S_z, S_w\}$, where $S_f = P_{H \ominus M}M_f|_{H \ominus M}$ for any polynomial $f$.

A submodule $M$ is called quasi-homogeneous if for any $h \in M$ and letting $h = \sum l h_l$ be the quasi-homogeneous decomposition with respect to (1.2), we have $h_l \in M$. An equivalent definition is that $M$ is generated by some quasi-homogeneous polynomials $p_1, \ldots, p_s$ with degrees $l_1, \ldots, l_s$, respectively. In this case, $M, H \ominus M$ can be decomposed as

$$M = \bigoplus M_l, \quad H \ominus M = \bigoplus (H_l \ominus M_l),$$

where $M_l = H_l \cap M = p_1 H_{l-l_1} + \cdots + p_s H_{l-l_s}$.

Our interest is in essentially normal Hilbert modules. An analytic Hilbert module $H$ is called essentially normal [Ar4, Ar5] if the tuple $\{M_z, M_w\}$ satisfies that both commutators $[M_z^*, M_z]$ and $[M_w^*, M_w]$ are compact. If $H$ is essentially normal, the Fuglede-Putnam theorem shows that $[M_z^*, M_w]$ is compact. Moreover, $H$ is $p$-essentially normal if $[M_z^*, M_z], [M_w^*, M_w]$ and $[M_z^*, M_w]$ belong to the Schatten-von Neumann class $\mathcal{L}^p$, where $p \in [1, \infty]$, and where $\mathcal{L}^\infty$ is interpreted as the compact operator ideal $\mathcal{K}$ [Ar5].

Essentially all of the basic Hilbert modules that have received attention over the years are essentially normal–including the reproducing function space defined by the kernel

$$K_v^{(\alpha, \beta)}(z, w) = \frac{1}{(1 - ((z, w), (\alpha, \beta))^{v}} \quad (v > 0).$$

When $v = 1$, the module is the 2-shift Hilbert module [Ar1, Ar2]; when $v = 2$, the module is the usual Hardy module $H^2(\mathbb{B}_2)$; and when $v = 3$, the module is the usual Bergman module $L^2_\alpha(\mathbb{B}_2)$. All the above modules have the following property:

$$\sigma(M_z, M_w) = \mathbb{B}_2, \quad \sigma_e(M_z, M_w) = \partial \mathbb{B}_2.$$

From [GHX], these results on spectrum are equivalent to the condition

$$\lim_{n} \frac{a_n}{a_{n+1}} = 1. \quad (1.4)$$

In this paper, we always assume that the analytic Hilbert module $H$ defined by $f(z) = \sum a_n z^n$ satisfies (1.1) and (1.4), that is, the following hold

(1). $\sum a_n = \infty$; \hspace{1cm} (2). $\lim_{n} \frac{a_n}{a_{n+1}} = 1. \quad (1.5)$
For an essentially normal Hilbert module, one can ask which submodules
and quotient submodules are essentially normal. Arveson and Douglas have
noticed that if $H$ is essentially normal, then either both a submodule $M$ and
the corresponding quotient module $H/M$ are essentially normal or neither
are [Ar4, Dou2]. If a submodule $M$ of $H$ is essentially normal, then the
$C^*$-algebra $C^*(M^⊥)$ generated by $\{I, S_z, S_w\}$ is essentially commutative,
and it contains the compact operator ideal $K$, and hence yields an extension
\[0 \rightarrow K \hookrightarrow C^*(M^⊥) \xrightarrow{\pi} C(\sigma_e(S_z, S_w)) \rightarrow 0.\] (1.6)
This $C^*$-extension gives rise to an $K$-homology element for the essential
spectrum of the tuple $\{S_z, S_w\}$, which is an invariant for the quotient module
[BDF1, BDF2, DP]. Hence, Arveson conjectured that all graded submodules
for $d$-shift Hilbert module with finite multiplicity are essentially normal[Ar3,
Ar4], and he established essentially normality in the case the submodule is
generated by monomials. Douglas generalized this result to the cases in
which the $d$-shift is replaced by more general weighted shifts [Dou2]. In
dimension $d = 2$, a result of Guo [Guo2] implies an affirmative answer for
Arveson’s conjecture, and that result was generalized by Guo and Wang
[GW]. In [Ar5], Arveson reduced the problem to the case of linearity with
finite multiplicity.

This paper mainly concerns quasi-homogeneous submodules. In section
2 it is shown that if $M$ is a quasi-homogeneous submodule of a $U$-invariant
analytic Hilbert module $H$ on $\mathbb{B}_2$ which satisfies the condition (1.5), then $M$
is essentially normal. In section 3, we show that each quasi-homogeneous
submodule of $L^2_0(\mathbb{B}_2)$ is $p$-essentially normal for $p > 2$, and the method is
also valid for the Hardy module and the weighted Bergman modules. Section
4 is associated with $K$-homology invariants arising from quasi-homogeneous
quotient modules.

2 Essential normality

In this section we mainly consider the $U$-invariant reproducing function
space $H$ on $\mathbb{B}_2$ which satisfies the condition (1.5). From [GRS] it follows
that the tuple $\{M_z, M_w\}$ is essentially normal and $\sigma_e(M_z, M_w) = \partial \mathbb{B}_2$. Let
$C^*\{M_z, M_w\}$ denote the $C^*$-algebra generated by the identity operator $I$
and the coordinate multipliers $\{M_z, M_w\}$. Then one has a $C^*$-extension
\[0 \rightarrow K \hookrightarrow C^*\{M_z, M_w\} \xrightarrow{\pi} C(\partial \mathbb{B}_2) \rightarrow 0,\] (2.1)
where $\pi(M_z) = z$ and $\pi(M_w) = w.
Concerning a submodule $M$ of $H$, the two tuples $\{R_z, R_w\}$, $\{S_z, S_w\}$ acting on $M, H \ominus M$, respectively, are associated intimately, where

$$R_f = M_f|_M, \quad S_f = P_{H \ominus M}M_f|_{H \ominus M}$$

for any polynomial $f$. The next lemma, first appeared in [Ar4, Dou2], gives a connection between the two tuples.

**Lemma 2.1.** Given an operator $T$ on $H$ and an invariant subspace $M$ of $T$, then $T$ can be decomposed as

$$T = \begin{bmatrix} R & C \\ 0 & S \end{bmatrix},$$

corresponding to $H = M \oplus M^\perp$. If $[T^*, T] \in \mathcal{L}^p$, then the following conditions are equivalent:

1. $[R^*, R] = R^*R - RR^* \in \mathcal{L}^p$,
2. $[S^*, S] = S^*S - SS^* \in \mathcal{L}^p$,
3. $C = P_{M}TP_{M^\perp} \in \mathcal{L}^{2p}$,

where $p \in [1, \infty]$, and $\mathcal{L}^\infty$ is interpreted as the compact operator ideal $K$.

The above lemma plays an important role in this paper. Moreover, we require the following lemma.

**Lemma 2.2.** Setting $A = C^*\{f_q \mid f \in \mathbb{C}[z, w]\} + \mathcal{K} \subset C^*\{M_z, M_w\}$, where $q = (z^n - \alpha \omega^m)^l$ with $\alpha \neq 0$, then both $M_zM_z^* - t^2$ and $M_wM_w^* - (1 - t^2)$ are in $A$ for some positive number $t$.

**Proof.** Notice that $A$ is a subalgebra of the $C^*$-algebra $C^*\{M_z, M_w\}$. Considering (2.1), one sees that

$$\pi(A) = C^*\{f_q \mid f \in \mathbb{C}[z, w]\},$$

which is a $C^*$-subalgebra of $C(\partial \mathbb{B}_2)$. By [Ped, p146], a well known version of the Stone-Weierstrass Theorem states that given a locally compact Hausdorff space $X$, and a self-adjoint, point-separating subalgebra $Y$ of $C_0(X)$ that does not vanish identically at any point, then $Y$ is uniformly dense in $C_0(X)$. This means that $\pi(A)$ coincides with $C_0(\partial \mathbb{B}_2 \setminus Z(q))$. Since $\ker \pi = \mathcal{K} \subseteq A$, we have

$$A = \pi^{-1}(\pi(A)) = \pi^{-1}(C_0(\partial \mathbb{B}_2 \setminus Z(q))).$$

It remains to show that $\pi(M_zM_z^* - t^2) = |z|^2 - t^2$ is identically vanished on $\partial \mathbb{B}_2 \cap Z(q)$ for some $t$. Since the function $f : [0, \infty) \to (-\infty, \infty)$ defined by

$$f(x) = x^2 + |\alpha|^{\frac{2}{m}}x^{\frac{2n}{m}} - 1$$

for some positive numbers $n$ and $m$.
is strictly increasing, and \( f(0) < 0, f(1) > 0 \), there exists a unique positive \( t_0 \) satisfying \( f(t_0) = 0 \).

For any \((z, w) \in \partial B_2 \cap Z(q)\), it satisfies
\[
|z|^2 + |w|^2 = 1, \quad z^n = \alpha w^m.
\]
Hence
\[
|z|^2 + |\alpha|^{-\frac{2}{m}}|z|^\frac{2m}{m} = 1,
\]
that is, \( f(|z|) = 0 \). By the above reasoning, one has \(|z| = t_0\) and hence
\(|w|^2 = 1 - t_0^2\) for any \((z, w) \in \partial B_2 \cap Z(q)\). This implies that both \(M_zM_z^* - t_0^2\) and \(M_wM_w^* - (1 - t_0^2)\) are in \(A\). This completes the proof of lemma 2.2. \(\Box\)

By [Ar4, Dou2], if \(M = [q]\) is the submodule generated by \(q\), where \(q = z^s\) or \(q = w^s\) for some positive integer \(s\), then \(M\) is essentially normal. Now we consider the case \(q = (z^n - \alpha w^m)^s\) with \(\alpha \neq 0\).

**Proposition 2.3.** If \(M = [q]\) is the quasi-homogeneous submodule of \(H\) generated by \(q = (z^n - \alpha w^m)^s\), then \(M\) is essentially normal.

**Proof.** Let \(f\) be a polynomial, and \(h_1, h_2 \in M^\perp\), then
\[
\langle S_qf h_1, h_2 \rangle = \langle qf h_1, h_2 \rangle = 0.
\]
Hence \(S_qf = 0\). Using Lemma 2.1, \(P_M M_qf P_{M^\perp} \in K\). This means that \(M_qf \in \{P_M\}_e\), the essential commutator of \(P_M\). Therefore,
\[
C^*\{M_qf | f \in C[z, w]\} + K \subseteq \{P_M\}_e.
\]

By Lemma 2.2, there exists a positive number \(t\) such that
\[
M_zM_z^* - t^2 \in C^*\{M_qf | f \in C[z, w]\} + K,
\]
hence
\[
M_zM_z^* = (M_zM_z^* - t^2) + t^2 \in \{P_M\}_e.
\]

Decompose \(M_q\) and \(M_z\) as follows,
\[
M_q = \begin{bmatrix} R_q & C_q \\ 0 & 0 \end{bmatrix}, \quad M_z = \begin{bmatrix} R_z & C_z \\ 0 & S_z \end{bmatrix}
\]
corresponding to \(H = M \oplus M^\perp\). An easy matrix calculation shows that
\[
M_qM_q^* + M_zM_z^* = \begin{bmatrix} R_qR_q^* + C_qC_q^* + R_zR_z^* + C_zC_z^* & C_zS_z^* \\ S_zC_z^* & S_zS_z^* \end{bmatrix}. \quad (2.2)
\]
Since \( M_q M_q^* + M_z M_z^* \in \{ P_M \} \), that is, the commutant
\[
[M_q M_q^* + M_z M_z^*, P_M] = \begin{bmatrix}
0 & -C_z S_z^* \\
S_z C_z^* & 0
\end{bmatrix}
\]
is compact, this insures that \( C_z S_z^* \) is compact. Since \(|q|^2 + |z|^2 \) has no zero point on \( \partial \mathbb{B}_2 \), \( M_q M_q^* + M_z M_z^* \) is Fredholm. It follows from the decomposition (2.2) that \( S_z S_z^* \) is Fredholm. This implies that the range of \( S_z \) is of finite codimension.

We claim that \( S_z \) is Fredholm. It suffices to show that \( \ker S_z \) is of finite dimension. Given any \( h \in \ker S_z \), letting \( h = \sum_l h_l \) be the quasi-homogeneous decomposition with respect to (1.3), we have
\[
zh = \sum_l zh_l \in M.
\]
Since \( M \) is quasi-homogeneous,
\[
zh_l \in M_l = q \mathbb{H}_{l-mns+m}.
\]
Noticing \( \text{GCD}(z, q) = 1 \), one sees that \( h_l \in M \) and hence \( h = 0 \). This implies that \( \ker S_z = \{ 0 \} \) and it follows that \( S_z \) is Fredholm.

Combining the above with the fact \( C_z S_z^* \) is compact, we see that \( C_z \) is compact and hence \( R_z \) is essentially normal by Lemma 2.1. Using the same argument, \( R_w \) is essentially normal and hence \( M \) is essentially normal. This completes the proof of proposition 2.3.

To continue, we need a technical lemma.

\textbf{Lemma 2.4.} Given two polynomials \( f, g \), assume \( N = [f] \) and \( L = [g] \) are \( p \)-essentially normal. If \( Z(f) \cap Z(g) \cap \partial \mathbb{B}_2 = \phi \), then \( M = [fg] \) is \( p \)-essentially normal.

\textbf{Proof.} Let \( h_1 \in M^\perp \), \( h_2 \in H \), then
\[
\langle M_g h_1, fh_2 \rangle = \langle h_1, gfh_2 \rangle = 0.
\]
This implies \( M_g M^\perp \subseteq N^\perp \). Since \( N \) is \( p \)-essentially normal, and \( P_N \geq P_M \), Lemma 2.1 implies that for any polynomial \( h \),
\[
P_M M_h M_g M_g^* P_M^\perp = P_M (P_N M_h P_N^\perp) M_g^* P_M^\perp \in \mathcal{L}^{2p}.
\] (2.3)
Using the same reasoning, one has that
\[
P_M M_h M_f M_f^* P_M^\perp \in \mathcal{L}^{2p}.
\] (2.4)
Taking $h = 1$, we see that

$$P_M(M_f M_f^* + M_g M_g^*)P_M \perp \in \mathcal{L}^{2p}. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) shows that for any polynomial $h$,

$$P_M M_h P_M \perp (M_f M_f^* + M_g M_g^*) P_M \perp = P_M M_h P_M (M_f M_f^* + M_g M_g^*) P_M \perp \in \mathcal{L}^{2p}. \tag{2.5}$$

Since $Z(f) \cap Z(g) \cap \partial \mathbb{B}_2 = \emptyset$, by the extension (2.1) $M_f M_f^* + M_g M_g^*$ is Fredholm. It follows from (2.5) that the positive operator

$$P_M \perp (M_f M_f^* + M_g M_g^*) P_M \perp : M^\perp \rightarrow M^\perp$$

is Fredholm. This implies that $P_M M_h P_M \perp$ belongs to $\mathcal{L}^{2p}$ for any polynomial $h$. The desired result comes from Lemma 2.1.

For any quasi-homogeneous polynomial $p$, it can be uniquely decomposed as

$$p(z, w) = \alpha z^w w^\sigma \prod_{i=1}^{l} (z^n - \alpha_i w^m)^{s_i}.$$ 

Since $Z(z^n - \alpha w^m) \cap Z(z^n - \beta w^m) \cap \partial \mathbb{B}_2 = \emptyset$ if $\alpha \neq \beta$, using Proposition 2.3 and Lemma 2.4 shows that the submodule $[p]$ is essentially normal.

Given any quasi-homogeneous submodule $M$, then $I = M \cap \mathbb{C}[z, w]$ is a quasi-homogeneous ideal, and from [CG, p28] $I$ has the Beurling form $I = pL$, where $p = \text{GCD}\{q : q \in I\}$ is quasi-homogeneous and $L$ is a finite codimensional ideal of $\mathbb{C}[z_1, z_2]$ by [Yang, Lemma 6.1]. This means that there is a finite dimensional space $R$ such that $\mathbb{C}[z_1, z_2] = L + R$, and hence

$$p \mathbb{C}[z_1, z_2] = pL + pR.$$ 

Thus $M$ is a finite codimensional submodule of $[p]$. It follows that $M$ is essentially normal.

The above discussion gives the following theorem.

**Theorem 2.5.** If $M$ is a quasi-homogeneous submodule of $H$, then $M$ is essentially normal.

### 3 $p$-essential normality

It is well known that the usual Bergman module $L_a^2(\mathbb{B}_d)$ are $p$-essentially normal for $p > d$. In [GW], it was proved that each homogeneous submodule
of the Bergman module $L^2_a(\mathbb{B}_2)$ over the two dimensional unit ball $\mathbb{B}_2$ is $p$-essentially normal for $p > 2$. In this section we will show if $M$ is a quasi-homogeneous submodule of $L^2_a(\mathbb{B}_2)$, then $M$ is $p$-essentially normal for $p > 2$. However, the method in [GW] is not valid in the case of quasi-homogeneous submodules and new techniques are needed.

Let $P$ be the orthogonal projection from $L^2(\mathbb{B}_2)$ onto $L^2_a(\mathbb{B}_2)$. The Toeplitz operator $T_f : L^2_a(\mathbb{B}_2) \to L^2_a(\mathbb{B}_2)$ with symbol $f \in L^\infty(\mathbb{B}_2)$ is defined by $T_f(h) = P(fh)$ for all $h \in L^2_a(\mathbb{B}_2)$. The Hankel operator $H_f$ with symbol $f$ is defined by $H_fh = (I - P)(fh)$ for all $h \in L^2_a(\mathbb{B}_2)$. For $f, g \in L^\infty(\mathbb{B}_2)$, Toeplitz and Hankel operators are connected by the following formula

$$T_fg - T_fT_g = H^*_f \bar{g}H_g.$$  

(3.1)

It is easy to verify that for any polynomial $g$

$$H^*_g \bar{g} = T^*_g - T_gT^*_g \in L^p$$  

(3.2)

that is, $H_g \in L^{2p}$ for any $p > 2$. Another result states that if $f$ is compactly supported, then $T_f \in L^1$. This is a well known fact, but we couldn’t find a precise reference, and hence give the detail of the proof.

**Lemma 3.1.** If $f = g\chi_E$, where $g \in L^\infty(\mathbb{B}_2)$ and $\chi_E$ is the characteristic function of a compact subset $E$ of $\mathbb{B}_2$, then $T_f \in L^1$.

**Proof.** By [Zhu, p19], it suffices to show that there exists a constant $C$ such that for any orthonormal basis $\{e_n\}$, $\sum_n |\langle T_fe_n, e_n \rangle| < C$. Since for each $(\alpha, \beta) \in \mathbb{B}_2$,

$$K_{(\alpha, \beta)}(z, w) = \frac{1}{(1 - \bar{\alpha}z - \bar{\beta}w)^3} = \sum_n e_n(z, w)e_n(\alpha, \beta),$$

this implies

$$\sum_n |\langle T_fe_n, e_n \rangle| \leq \sum_n \int_{|z|^2 + |w|^2 < 1} |f(z, w)||e_n(z, w)|^2dA$$

$$\leq \int_{|z|^2 + |w|^2 < 1} |f(z, w)||K_{(z, w)}(z, w)dA$$

$$= \int_E \frac{|g(z, w)|}{(1 - |z|^2 - |w|^2)^3}dA,$$

where $dA$ is the area measure. Since $g$ is bounded and $E$ is a compact subset of $\mathbb{B}_2$, the last integral is finite. This means that $T_f \in L^1$, as desired. \qed

Moreover, we need the following lemma.
Lemma 3.2. Let $M = [q^s]$ be a submodule generated by $q^s$, where $s$ is a positive integer and $q(z, w) = z^n - \alpha w^m$. Then $[T_{fq}, P_M] \in \mathcal{L}^{2p}$ for any $f \in L^\infty(\mathbb{B}_2)$ and $p > 2$.

Proof. We claim that $T_q P_M \perp \in \mathcal{L}^{2p}$ for $p > 2$. The proof is given by induction. Firstly, in the case $s = 1$, we have $S_q = P_M T_q P_M \perp = 0$. Using Lemma 2.1 (3) shows that $T_q P_M \perp \in \mathcal{L}^{2p}$ for $p > 2$. Now assume that the claim is true in the case $s = k - 1$, that is, letting $L = [q^{k-1}]$, then $T_q P_L \perp \in \mathcal{L}^{2p}$ for $p > 2$. By Lemma 2.1 it follows that $R_q = T_q|_L$ satisfies that $[R_q, R_q^*] \in \mathcal{L}^p$ for $p > 2$. Viewing $M = [q^k]$ as a submodule of $L$, since $qL \subseteq M$, $R_q$ can be decomposed as $R_q = \begin{bmatrix} R_q^{(1)} & C_q \\ 0 & 0 \end{bmatrix}$ with respect to $L = M \oplus (L \ominus M)$, where $R_q^{(1)} = T_q|_M$ and $C_q = P_M T_q|_{L \ominus M}$. Since $[R_q, R_q^*] \in \mathcal{L}^p$ for $p > 2$, applying Lemma 2.1 shows that $C_q = T_q P_{L \ominus M} \in \mathcal{L}^{2p}$ for $p > 2$. Therefore, $T_q P_M \perp = T_q P_L \perp + T_q P_{L \ominus M} \in \mathcal{L}^{2p}$. This implies that the claim is true. From the claim we obtain that $P_M T_q = P_M T_q P_M \perp \in \mathcal{L}^{2p}$. (3.4)

Thus, by (3.2), (3.3) and (3.4), for any $f \in L^\infty(\mathbb{B}_2)$

$$[T_{fq}, P_M \perp] = T_{fq} P_M \perp - P_M \perp T_{fq} = T_{fq} T_q P_{M \perp} - (P_M \perp T_q T_f + P_{M \perp} H_q H_f) \in \mathcal{L}^{2p}$$

for $p > 2$. This completes the proof of Lemma 3.2. □

Theorem 3.3. If $M$ is a quasi-homogeneous submodule of $L^2_a(\mathbb{B}_2)$, then $M$ is $p$-essentially normal for $p > 2$.

Proof. Firstly, we consider the case $M = [(z^n - \alpha w^m)^s]$. Let $E = \{(z, w) : \frac{1}{2} < |z|^2 + |w|^2 < 1\}$
be the subset of $B^2$ and then $E_c = \{(z, w) : |z|^2 + |w|^2 \leq \frac{1}{2}\}$. Considering the function $\phi(z, w)$ defined by

$$\phi(z, w) = |z|^{\frac{2n}{m}} - |\alpha|^{\frac{2}{m}} |w|^2,$$

we see that

$$\left| \frac{\phi(z, w)}{z^n - \alpha w^m} \right| = \left| \frac{|z|^{\frac{2n}{m}} - |\alpha|^{\frac{2}{m}} |w|^2}{z^n - \alpha w^m} \right| \leq \left| \frac{|z|^{\frac{2n}{m}} - |\alpha|^{\frac{2}{m}} |w|^2}{|z|^n - |\alpha||w|^m} \right| = \frac{|z|^{\frac{n}{m}} + |\alpha|^{\frac{1}{m}} |w|}{\sum_{i=0}^{m-1} |z|^{\frac{m}{m}} (|\alpha|^{\frac{1}{m}} |w|)^{m-1-i}}$$

is bounded on $E$. Setting $F(z, w) = \phi(z, w) / (z^n - \alpha w^m)$, and hence

$$\phi(z, w) = F(z, w) (z^n - \alpha w^m).$$

By Lemma 3.2,

$$[T_{g\phi} I_{E_c}, P_M] \in \mathcal{L}^{2p}$$

for any $g \in L^\infty$ and $p > 2$. Applying Lemma 3.1, we have that

$$T_{g\phi I_{E_c}} \in \mathcal{L}^1, \quad g \in L^\infty.$$

This implies that

$$[T_{g\phi}, P_M] \in \mathcal{L}^{2p}. \quad (3.5)$$

Setting $\varphi(z, w) = |z|^2 + |w|^2 - 1$, an easy calculation shows that

$$T_{\varphi} z^i w^j = \frac{-1}{i + j + 3} z^i w^j$$

and hence $T_{\varphi} \in \mathcal{L}^p$ for $p > 2$. Moreover, since

$$H_{\varphi} = H z T_z + H w T_w \in \mathcal{L}^{2p},$$

one sees that for any $g \in L^\infty$,

$$T_{g\varphi} = T_g T_{\varphi} + H^*_z H_{\varphi} \in \mathcal{L}^{2p} \quad (3.6)$$

for $p > 2$. Put

$$\psi(z, w) = \varphi(z, w) + |\alpha|^{\frac{2}{m}} \phi(z, w) = |z|^2 + |\alpha|^{\frac{2}{m}} |z|^{\frac{2n}{m}} - 1.$$
Combining (3.5) with (3.6), for any $g \in L^\infty(\mathbb{D})$, it holds that
\[ [T_{g\psi}, P_M] \in \mathcal{L}^{2p}, \] (3.7)

Considering the polynomial $f(t) = t^{2m} + |\alpha|^{\frac{2}{m}} t^{2n} - 1$, by the argument in Lemma 2.2, there exists a unique positive root, denoted by $r$ such that
\[ f(t) = c (t - r) \prod_i (t - \beta_i), \]
where $c$ is a constant, $\beta_i$ are the other roots of $f(t)$ and $\beta_i \notin [0, \infty)$. This implies that
\[ \frac{|z|^2 - r^{2m}}{\psi(z, w)} = \frac{|z|^2 - r^{2m}}{f(|z|^{\frac{1}{m}})} = \frac{|z|^2 - r^{2m}}{c (|z|^{\frac{1}{m}} - r) \prod_i (|z|^{\frac{1}{m}} - \beta_i)} = \sum_{i=0}^{2m-1} \frac{|z|^{\frac{1}{m}} r^{2m-i-1}}{c \prod_i (|z|^{\frac{1}{m}} - \beta_i)} \]
is bounded on $\mathbb{B}_2$. It follows from (3.7) that
\[ [T_{|z|^2}, P_M] = [T_{|z|^2 - r^{2m}}, P_M] \in \mathcal{L}^{2p}, \]
for $p > 2$. Since $T_{|z|^2} - T_z T_z^* = H_z^* H_z \in \mathcal{L}^p$ for $p > 2$, we have
\[ [T_z, T_z^*, P_M] \in \mathcal{L}^{2p} \]
for $p > 2$. Using the similar argument as in the proof of Proposition 2.3 shows that
\[ [T_z, P_M] \in \mathcal{L}^{2p}. \]
In the same way,
\[ [T_w, P_M] \in \mathcal{L}^{2p} \]
and hence $[T_f, P_M] \in \mathcal{L}^{2p}$ for any polynomial $f$ and $p > 2$. This implies that the submodule $M = [(z^n - \alpha w^m)^s]$ is $p$-essentially normal for $p > 2$.

Given a quasi-homogeneous polynomial $q$, $q$ can be decomposed as
\[ q(z, w) = \alpha z^u w^s \prod_{i=1}^l \left( (z^n - \alpha_i w^m)^{s_i} \right). \]
Using lemma 2.4, \([q]\) is \(p\)-essentially normal for \(p > 2\).

In general, if \(M\) is quasi-homogeneous submodule, then \(M\) has the form \(M = \langle qL \rangle\), where \(q\) is quasi-homogeneous polynomial and \(L\) is a quasi-homogeneous ideal, and \(\dim \mathbb{C}[z, w]/L < \infty\). Using the same argument as in Theorem 2.5, we have \(M\) is \(p\)-essentially normal for \(p > 2\). This completes the proof of Theorem 3.3. \(\square\)

It is clear that the method in Theorem 3.3 is also valid in the case of the Hardy module and the weighted Bergman modules over \(\mathbb{B}_2\).

Unfortunately, since the above method strongly depends on the measure theory, it can’t be generalized to many other spaces, such as the 2-shift Hilbert module over \(\mathbb{B}_2\). Recall the 2-shift Hilbert module \(H_2^2\) is the function space with the reproducing kernel

\[
K_{(\alpha, \beta)}(z, w) = \frac{1}{1 - \langle(z, w), (\alpha, \beta)\rangle}.
\]

In the next example, we show that if \(M = [z^n - \alpha w^m]\) is a submodule of the 2-shift Hilbert module \(H_2^2\) over \(\mathbb{B}_2\), then \(M\) is \(p\)-essentially normal for \(p > 2\) and \(H_2^2 \ominus M\) is 1-essentially normal. This approach may be useful to solve the general problem.

**Example 1.** Consider the submodule \(M = [z^n - \alpha w^m]\) of the 2-shift Hilbert module \(H_2^2\). Without loss of generality, suppose \(\alpha > 0\). It is easy to verify that

\[
Z(M) \cap \mathbb{B}_2 = \{(\alpha^\frac{1}{n} \lambda^m, \lambda^n) : \alpha^\frac{2}{n} |\lambda|^2m + |\lambda|^2n < 1\},
\]

where \(Z(M) = \{(z, w) \in \mathbb{C}^2 : z^n - \alpha w^m = 0\}\). Indeed, it suffices to show \(Z(M) \subseteq \{(\alpha^\frac{1}{n} \lambda^m, \lambda^n) : \lambda \in \mathbb{C}\}\). Given \((z_0, w_0) \in Z(M)\), choosing \(t\) such that \(t^n = w_0\), it follows that

\[
z_0^n = \alpha w_0^m = \alpha t^{mn}.
\]

Hence

\[
z_0 = \alpha^\frac{1}{n} t^m e^{\frac{2k\pi}{n}\sqrt{-1}}
\]

for some integer \(k\). Since \(GCD(m, n) = 1\), there are two integers \(k_1, k_2\) such that \(k_1m + k_2n = k\). Thus,

\[
z_0 = \alpha^\frac{1}{n} t^m e^{\frac{2k\pi}{n}\sqrt{-1}} = \alpha^\frac{1}{n} t^m e^{\frac{2m(k_1)}{n}\pi \sqrt{-1}} = (te^{\frac{2k_1\pi}{n}\sqrt{-1}})^m,
\]

and

\[
w_0 = t^n = (te^{\frac{2k_1\pi}{n}\sqrt{-1}})^n.
\]
Therefore, by setting $\lambda = t e^{\frac{2k_j}{\pi} \sqrt{-1}}$, we have

$$(z_0, w_0) = (\alpha \frac{i}{n} \lambda^m, \lambda^n),$$

as desired.

For $(\alpha \frac{i}{n} \lambda^m, \lambda^n) \in \mathbb{B}_2$, let

$$K_{(\alpha \frac{i}{n} \lambda^m, \lambda^n)}(z, w) = \frac{1}{1 - \alpha \frac{i}{n} \lambda^m \bar{z} - \bar{\lambda}^n w} = \sum_{l=0}^{\infty} G_l(z, w) \bar{\lambda}^l$$

be the Taylor expansion in the variable $\bar{\lambda}$. An easy calculation shows that

$$G_l(z, w) = \sum_{mi + nj = l} \frac{(i + j)!}{i! j!} \alpha \frac{i}{n} z^i w^j,$$

and then $G_l \in \mathbb{H}_l$. In particular, $G_0 = 1 \in M^\perp$. Since

$$\frac{K_{(\alpha \frac{i}{n} \lambda^m, \lambda^n)} - 1}{\lambda} = \sum_{l=1}^{\infty} G_l \bar{\lambda}^{l-1} \rightarrow G_1$$

in the norm of $H^2_2$ as $\lambda \rightarrow 0$, we have $G_1 \in M^\perp$. In the same way, one sees that $G_l \in M^\perp$ for any $l$. Since $\{K_{(\alpha \frac{i}{n} \lambda^m, \lambda^n)}\}$ is dense in $M^\perp$,

$$M^\perp = \text{span}\{G_l\}.$$ 

Since $M$ is quasi-homogeneous, $M^\perp$ can be decomposed as

$$M^\perp = \bigoplus_l (\mathbb{H}_l \cap M^\perp).$$

Moreover,

$$\mathbb{H}_l \cap M^\perp = \mathbb{C} G_l.$$ 

A direct calculation shows that

$$S^*_z G_l = \alpha \frac{i}{n} G_{l-m},$$

and thus

$$S^*_z G_l = \alpha \frac{i}{n} \frac{\|G_l\|^2}{\|G_{l+m}\|^2} G_{l+m}.$$ 

This means that

$$(S^*_z S^*_z - S^*_z S_z) G_l = \alpha \frac{i}{n} \frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_l\|^2}{\|G_{l+m}\|^2} G_l.$$ 

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Now consider the polynomial \( f(x) = 1 - \alpha^2 x^m - x^n \). By the similar argument as in Lemma 2.2, \( f(x) \) can be decomposed as
\[
f(x) = c(r - x) \prod_i (b_i - x)^{s_i},
\]
where \( r \) is the unique positive root of \( f(x) \), \( b_i \) are the other roots of \( f(x) \) and \( b_i \notin [0, \infty) \) for each \( i \). We claim that \( r < |b_i| \) for each \( i \). Otherwise, if \( r \geq |b_i| \) for some \( i \), we have
\[
1 = \alpha^2 r^m + r^n \geq |\alpha^2 b_i^m| + |b_i^n| \geq |\alpha^2 b_i^m + b_i^n| = 1 = \alpha^2 b_i^m + b_i^n.
\]
This implies that \( b_i^m = r^m, b_i^n = r^n \). Since \( \text{GCD}(m, n) = 1 \), there are integers \( u, v \) such that \( mu + nv = 1 \). It follows that
\[
b_i = (b_i^m)^u (b_i^n)^v = (r^m)^u (r^n)^v = r.
\]
This leads to a contradiction. Therefore, \( r < \min_i |b_i| \). Since
\[
\|K_{(\alpha^\frac{1}{2} \lambda^m, \lambda^n)}\|^2 = \frac{1}{1 - |\alpha^\frac{1}{2} \lambda^m|^2 - |\lambda^n|^2}
\]
\[
= \frac{1}{c (r - |\lambda|^2) \prod_i (|b_i|^2)^{s_i}}
\]
\[
= \frac{C_r}{r - |\lambda|^2} + \sum_i s_i \sum_{j=1}^{s_i} \frac{C_{ij}}{b_i - |\lambda|^2}^j
\]
\[
= \sum_l \left[ \frac{C_r}{r^{l+1}} + \sum_i s_i \sum_{j=1}^{s_i} (l + 1) \cdots (l + j - 1) \frac{C_{ij}}{b_i^{l+j}} \right] |\lambda|^{2l},
\]
where \( C_r, C_{ij} \) are constants, and
\[
\|K_{(\alpha^\frac{1}{2} \lambda^m, \lambda^n)}\|^2 = \sum_l \|G_l\|^2 |\lambda|^{2l},
\]
we have
\[
\|G_l\|^2 = \frac{C_r}{r^{l+1}} + \sum_i s_i \sum_{j=1}^{s_i} (l + 1) \cdots (l + j - 1) \frac{C_{ij}}{b_i^{l+j}}
\]
Since \( r < \min_i |b_i| \), it follows that
\[
\|G_l\|^2 \sim \frac{C_r}{r^{l+1}},
\]

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where \( a(l) \sim b(l) \) means that there exists positive real numbers \( c_1, c_2 \) such that \( c_1 < \frac{|a(l)|}{|b(l)|} < c_2 \) for sufficiently large \( l \). Moreover, since

\[
\frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_{l-m}\|^2}{\|G_l\|^2} = \left[ \sum_i \sum_{j=1}^{s_i} (l + 1 - m) \cdots (l + j - 1 - m) \frac{C_{ij}}{b_i^{l+j-m}} \right] \|G_{l-m}\|^2 \\
\leq r^{1+m} \sum_i \left[ \sum_{j=1}^{s_i} (l + 1) \cdots (l + j - 1) \frac{C_{ij}}{b_i} \right] (\frac{r}{b_i})^j,
\]

we have

\[
\sum_l \left| \frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_{l-m}\|^2}{\|G_l\|^2} \right| < \infty,
\]

where \( a(l) \preceq b(l) \) means that there exists a positive real number \( c \) such that \( \frac{|a(l)|}{|b(l)|} < c \) for sufficiently large \( l \). Furthermore, from

\[
\frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_l\|^2}{\|G_l\|^2} = \sum_i \sum_{j=1}^{s_i} (l + 1 - m) \cdots (l + j - 1 - m) \frac{C_{ij}}{b_i^{l+j-m}} \sum_i \sum_{j=1}^{s_i} (l + 1) \cdots (l + j - 1) \frac{C_{ij}}{b_i} \\
= \sum_i \sum_{j=1}^{s_i} [(l + 1 - m) \cdots (l + j - 1 - m) b_i^m - (l + 1) \cdots (l + j - 1) m] \frac{r C_{ij}}{C_i b_i} (\frac{r}{b_i})^j,
\]

we obtain that

\[
\sum_l \left| \frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_l\|^2}{\|G_l\|^2} \right| < \infty.
\]

It follows that

\[
\sum_l \left| \frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_l\|^2}{\|G_l\|^2} \right| \leq \sum_l \left| \frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_l\|^2}{\|G_l\|^2} \right| + \left| \frac{\|G_{l-m}\|^2}{\|G_l\|^2} - \frac{\|G_{l-m}\|^2}{\|G_l\|^2} \right| + \left| \frac{\|G_l\|^2}{\|G_l\|^2} - \frac{\|G_l\|^2}{\|G_l\|^2} \right| < \infty,
\]

and hence \([S_z, S_z^*] \in \mathcal{L}^1\). Using the same argument, both \([S_z, S_u^*] \] and \([S_u, S_u^*] \) are in \( \mathcal{L}^1 \). It follows that \( M \perp \) is \( 1 \)-essentially normal. Moreover, using Lemma 2.1 shows that \( M \) is \( p \)-essentially normal for \( p > 2 \), as desired. \( \square \)

Combining the proof of the above Example and Lemma 2.4, we have
Proposition 3.4. Let \( p(z, w) = z^a w^b (z^n - \alpha_1 w^m) \cdots (z^n - \alpha_i w^m) \), and \( \alpha_i \neq \alpha_j \) if \( i \neq j \). Then \([p]\), as a submodule of the 2-shift Hilbert module \( H^2_2 \), is \( p \)-essentially normal for \( p > 2 \), and the quotient \([p]_1^\perp\) is 1-essentially normal.

4 K-homology

Given a \( \mathcal{U} \)-invariant analytic Hilbert module \( H \) over \( \mathbb{B}_2 \), let \( M \) be a submodule of \( H \) and \( M^\perp \) the corresponding quotients module. It was shown that if \( M \subseteq \{ f : f(0) = 0 \} \), then the \( C^* \)-algebra \( C^*(M^\perp) \) generated by the tuple \( \{ I, S_z, S_w \} \) acting on \( M^\perp \) is irreducible in \([GW]\). Therefore, if \( M \) is a quasi-homogeneous submodule of \( H \) satisfying (1.5), from Theorem 2.5, the tuple \( \{ S_z, S_w \} \) is essentially normal, and hence \( C^*(M^\perp) \) contains all compact operators. This yields the extension

\[
0 \rightarrow \mathcal{K} \hookrightarrow C^*(M^\perp) \xrightarrow{\pi} C(\sigma_e(S_z, S_w)) \rightarrow 0,
\]

where \( \sigma_e(S_z, S_w) \) denotes the Taylor essential spectrum of the tuple \( \{ S_z, S_w \} \).

Below we will describe the Taylor essential spectrum of the tuple \( \{ S_z, S_w \} \).

In \([GW]\), for a homogeneous submodule \( M \), it is shown that

\[
\sigma_e(S_z, S_w) = Z(M) \cap \partial \mathbb{B}_2,
\]

where \( Z(M) \) is the zero variety of the ideal \( M \cap \mathbb{C}[z, w] \). If \( M \) is quasi-homogeneous with weight \((m, n)\), a notable fact is that if \((z, w) \in Z(M)\), then \((t^m z, t^n w) \in Z(M)\) for any complex number \( t \). Indeed, for any polynomial \( f \in \mathbb{H}_l \), it holds that \( f(t^m z, t^n w) = t^l f(z, w) \).

Theorem 4.1. If \( M \) is a quasi-homogeneous submodule of a \( \mathcal{U} \)-invariant analytic Hilbert module \( H \) on \( \mathbb{B}_2 \) which satisfies the conditions (1.5), then the tuple \( \{ S_z, S_w \} \) acting on \( M^\perp \) is essentially normal and

\[
\sigma(S_z, S_w) = Z(M) \cap \partial \mathbb{B}_2, \quad \sigma_e(S_z, S_w) = Z(M) \cap \partial \mathbb{B}_2.
\]

Therefore we have the extension

\[
0 \rightarrow \mathcal{K} \hookrightarrow C^*(M^\perp) \xrightarrow{\pi} C(Z(M) \cap \partial \mathbb{B}_2) \rightarrow 0.
\]

Proof. The essential normality comes from Theorem 2.5. For results on spectrum, the proof is similar to that of \([GW]\), but slightly different. For the reader’s convenience, we give the details of the proof.

Given \( \mu = (\mu_1, \mu_2) \in \mathbb{C}^2 \), \(|\mu| > 1\), and considering the function \( f_\mu(z, w) = \bar{\mu}_1 z + \bar{\mu}_2 w - |\mu|^2 \), from the proof of Theorem 4.5 in \([GHX]\), the multiplication operator \( M_{f_\mu} \) on \( H \) is invertible with the inverse \( M_{f_\mu}^{-1} \). A simple analysis
shows that $Rf_\mu$ acting on $M$ is invertible with the inverse $R_{\mu}^{-1}$ and hence $0 \notin \sigma(R_{f_\mu})$. Hence by the spectral mapping theorem [Tay], $\mu \notin \sigma(R_z, R_w)$. It follows that $\sigma(R_z, R_w) \subseteq \mathbb{B}_2$. By [Cur1,p39],

$$\sigma(S_z, S_w) \subseteq \mathbb{B}_2.$$ 

Since $M$ is essentially normal, Lemma 2.1 implies that $P_M M f P_{H \oplus M}$ is compact for any polynomial $f$. Therefore, by [Cur1,p39],

$$\sigma_c(S_z, S_w) \subseteq \sigma_c(M_z, M_w) \subseteq \partial \mathbb{B}_2.$$ 

Let $f \in M \cap \mathbb{C}[z, w]$ and $h_1, h_2 \in M^\perp$, then

$$\langle S f h_1, h_2 \rangle = \langle f h_1, h_2 \rangle = 0.$$ 

This implies that $S f = 0$ for any $f \in M \cap \mathbb{C}[z, w]$. By the spectral mapping theorem [Tay], we have

$$\sigma(S_z, S_w) \subseteq Z(M) \cap \mathbb{B}_2.$$ 

For $(z_0, w_0) \in Z(M) \cap \mathbb{B}_2$, we see that $K(z_0, w_0)$ is in $M^\perp$ and hence

$$(S_z^* - \bar{z}_0)K(z_0, w_0) = 0, \quad (S_w^* - \bar{w}_0)K(z_0, w_0) = 0.$$ 

This shows that $(z_0, w_0) \in \sigma(S_z, S_w)$ and thus $\sigma(S_z, S_w) \supseteq Z(M) \cap \mathbb{B}_2$. It follows that

$$\sigma(S_z, S_w) = Z(M) \cap \mathbb{B}_2.$$ 

It remains to show that

$$\sigma_c(S_z, S_w) \supseteq Z(M) \cap \partial \mathbb{B}_2.$$ 

Given any $(z_0, w_0) \in Z(M) \cap \partial \mathbb{B}_2$, it suffices to show that the operator $(S_z - z_0)(S_z - z_0)^* + (S_w - w_0)(S_w - w_0)^*$ is not Fredholm. Otherwise, there is a positive finite rank operator $F$ such that the operator

$$A = (S_z - z_0)(S_z - z_0)^* + (S_w - w_0)(S_w - w_0)^* + F$$

is positive and invertible. But on the other hand, since $(t^m z_0, t^n w_0) \in Z(M)$ for any $|t| < 1$, we have $K(t^m z_0, t^n w_0) \in M^\perp$ and it follows that

$$\langle ((S_z - z_0)(S_z - z_0)^* + (S_w - w_0)(S_w - w_0)^*)k(t^m z_0, t^n w_0), k(t^m z_0, t^n w_0) \rangle = |t^m z_0 - z_0|^2 + |t^n w_0 - w_0|^2 \to 0$$

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as \( t \to 1 \), where \( k_{(t^m z_0, t^n w_0)} = \frac{K_{(t^m z_0, t^n w_0)}}{\|K_{(t^m z_0, t^n w_0)}\|} \) is the normalized reproducing kernel. Moreover, since \( k_{(t^m z_0, t^n w_0)} \overset{w}{\to} 0 \) as \( t \to 1 \), we have that

\[
\|F k_{(t^m z_0, t^n w_0)}\| \to 0,
\]

and hence

\[
(A k_{(t^m z_0, t^n w_0)}, k_{(t^m z_0, t^n w_0)}) \to 0.
\]

This leads to a contradiction since the operator \( A \) is positive and invertible. Hence \((S_2 - z_0)(S_2 - z_0)^* + (S_w - w_0)(S_w - w_0)^*\) is not Fredholm. This implies that \((z_0, w_0) \in \sigma_\epsilon(S_2, S_w)\) if \((z_0, w_0) \in \mathcal{Z}(M) \cap \partial \mathbb{B}_2\). Therefore,

\[
\sigma(S_2, S_w) = \mathcal{Z}(M) \cap \bar{\mathbb{B}}_2, \quad \sigma_\epsilon(S_2, S_w) = \mathcal{Z}(M) \cap \partial \mathbb{B}_2.
\]

The proof is complete. \(\Box\)

To continue, we need some notations. Given a quasi-homogeneous ideal \( I \), if \([I]\), the submodule generated by \( I \), is essentially normal, then the corresponding quotient module also is, and hence the quotient module gives rise to an extension (4.3). From [BDF2], this extension gives a \( K \)-homology element in \( K_1(\mathcal{Z}(I) \cap \partial \mathbb{B}_2) \), which is denoted by \( e_I \). We will show that \( e_I \) is not trivial if \( I \) is a quasi-homogeneous ideal.

Let \( q \) be a quasi-homogeneous polynomial and \( M = [q] \). It is clear that \( \mathcal{Z}(M) = \mathcal{Z}(q) \). If \( q = (z^n - \alpha w^m)^\epsilon \) with \( \alpha \neq 0 \), choosing \( \beta \) such that \( \beta^m = \frac{1}{\alpha} \), from the discussion of Example 1 in Section 3, we have that

\[
\mathcal{Z}(M) = \{ (\lambda^m, \beta \lambda^n) : \lambda \in \mathbb{C} \}.
\]

Moreover, by the proof of Lemma 2.2, there is a positive number \( t_0 \) such that for any \((z, w) \in \mathcal{Z}(M) \cap \partial \mathbb{B}_2\), \(|z| = t_0\) and \(|w| = t_0^m \beta\). Letting \( \lambda_0 = \sqrt{t_0} \), then we have

\[
\mathcal{Z}(M) \cap \partial \mathbb{B}_2 = \{ (\lambda^m, \beta \lambda^n) : |\lambda| = \lambda_0 \}.
\]

It follows that the map \( \tau : \mathbb{T} \to \mathcal{Z}(M) \cap \partial \mathbb{B}_2 \) defined by

\[
\tau(t) = (\lambda_0^m t^m, \beta \lambda_0^n t^n)
\]

is homeomorphic, where \( \mathbb{T} \) is the boundary of the unit disc \( \mathbb{D} \). Since \( \text{GCD}(m, n) = 1 \), there exist positive integers \( k_1, k_2 \) such that \( mk_1 - nk_2 = 1 \). Noticing \(|w| = \lambda_0^n |\beta|\) if \((z, w) \in \mathcal{Z}(M) \cap \partial \mathbb{B}_2\), then it is easy to check that the inversion \( \tau^{-1} : \mathcal{Z}(M) \cap \partial \mathbb{B}_2 \to \mathbb{T} \) is given by

\[
\tau^{-1}(z, w) = \lambda_0^{-1} z^{k_1} \bigg( \frac{w}{\beta} \bigg)^{-k_2} = \lambda_0^{-1 - 2nk_2 - k_2} z^{k_1} \bar{w}^{k_2}.
\]
Furthermore, by [BDF1],
\[ K_1(Z(M) \cap \partial \mathbb{B}_2) = K_1(T) = \mathbb{Z}. \]

**Proposition 4.2.** If \( M = [(z^n - \alpha w^m)^s] \), then \( e_M \) is not trivial.

**Proof.** Using the above notations and noticing \(|w| = t^\beta n^\alpha \) on \( Z(M) \cap \partial \mathbb{B}_2 \), we have
\[ \pi(S_z S_{w^s}) = z^{k_1} w^{k_2} = \lambda_0^{1+2n k_2} \beta^{k_2} \tau^{-1}, \]
where \( \pi \) is given in (4.3). By [BDF1] and [AS], we have
\[ e_M = \text{ind}(S_z S_{w^s}) = k_1 \text{ind}S_z - k_2 \text{ind}S_w. \]

We claim that \( \text{ind}S_z = -ms \). In fact,
\[ M^\perp \ominus S_z M^\perp = H \ominus (S_z M^\perp + M) = H \ominus (z M^\perp + M) = H \ominus (zH + M) = H \ominus \mathbb{C}[z] + [(z^n - \alpha w^m)^s \mathbb{C}[z,w]] = H \ominus \mathbb{C}[z,w] + \mathbb{C}[z,w] = \{1, w, \ldots, w^{ms-1}\}. \]

Thus \( \dim \text{coker}(S_z) = ms \). From the proof of Proposition 2.3, one sees that \( \ker(S_z) = \{0\} \) and hence \( \text{ind}(S_z) = -ms \). Using the same argument, it is easy to show that \( \text{ind}(S_w) = -ns \). Since \( mk_1 - nk_2 = 1 \), we have
\[ e_M = k_1 \text{ind}S_z - k_2 \text{ind}S_w = -s \in K_1(Z(M) \cap \partial \mathbb{B}_2) \]
is not trivial, as desired. \( \square \)

To continue, we require a useful operation on \( K \)-homology, which is called the disjoint sum. Given two compact metric spaces \( X_1 \) and \( X_2 \) with \( X_1 \cap X_2 = \emptyset \), by [BDF1, Theorem 4.8], the disjoint sum gives the following isomorphism
\[ K_1(X_1 \bigsqcup X_2) \cong K_1(X_1) \oplus K_1(X_2), \]
where \( \bigsqcup \) denotes disjoint union of spaces.

The following proposition is slightly different from that in [GW], but the proof is similar, we omit it.
Proposition 4.3. Assume that $[I], [J]$ and $[IJ]$ are all essentially normal and let $[I], [J], [IJ]$ have the essential spectrum $Z(I) \cap \partial \mathbb{B}_2$, $Z(J) \cap \partial \mathbb{B}_2$ and $Z(IJ) \cap \partial \mathbb{B}_2$, respectively. If $Z(I) \cap Z(J) = \{0\}$, then

$$e_{IJ} = e_I \oplus e_J$$

in the sense of the disjoint sum.

For any quasi-homogeneous polynomial $p$, it can be decomposed as

$$p(z, w) = \alpha z^u w^v \prod_{i} (z^m - \alpha_i w^n)^{s_i},$$

such that

$$Z(z^m - \alpha_i w^n) \cap Z(z^m - \alpha_j w^n) = \{0\}, \text{ if } i \neq j.$$ Combining Propositions 4.2 with 4.3 gives the following.

Proposition 4.4. Under the above assumptions, we have

$$e_p = e_{z^u} \oplus e_{w^v} \cdots \oplus e_{(z^m - \alpha_i w^n)^s_i} \cdots = (-u, -v, \ldots, -s_i, \ldots).$$

Let $p$ and $q$ be quasi-homogeneous polynomials with the weight $(m, n)$. If they have the same zero set, then $p$ and $q$ can be decomposed as

$$p(z, w) = \alpha z^u w^v \prod_{i} (z^m - \alpha_i w^n)^{s_i},$$

$$q(z, w) = \alpha' z^{u'} w^{v'} \prod_{i} (z^m - \alpha_i w^n)^{s'_i},$$

From Proposition 4.4, we see that $e_p = e_q$ if and only if $p = c q$ for some constant $c$.

For each quasi-homogeneous ideal $I$ in $\mathbb{C}[z, w]$, $I$ can be uniquely written as the Beurling form $I = pL$ [Guo1], where $p = \text{GCD}\{q : q \in I\}$. It is easy to check that $p$ is quasi-homogeneous, and $L$ is a finite codimensional ideal of $\mathbb{C}[z, w]$ by [Yang, Lemma 6.1]. This ensures that $[p] = [I] \oplus N$ for some finite dimensional space $N$, and it follows that

$$[I]^\perp = [p]^\perp \oplus N.$$ Therefore, for submodules $[I]$ and $[p]$, the corresponding extensions (4.3) are weakly equivalent, and hence they are equivalent [BDF1, Theorem 1.8]. This means $e_p = e_I$. We summarize the above discussion in the following theorem, which is an analogue of a result in [GW]. This result is related to the conjecture made in [Dou3] relating the $K$-homology class to the fundamental class of the zero variety.
Theorem 4.5. Let $I = pL_1$ and $J = qL_2$ be the Beurling form of two quasi-homogeneous ideals $I$ and $J$. Then they determine two nontrivial elements $e_I \in K_1(Z(p) \cap \partial B_2)$, $e_J \in K_1(Z(q) \cap \partial B_2)$. Furthermore, if $p$ and $q$ have the same zero set, then $e_I = e_J$ if and only if $p = cq$ for some constant $c$.

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References


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