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Essentially normal Hilbert modules and K-homology

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Abstract This paper mainly concerns the essential normality of graded submodules. Essentially all of the basic Hilbert modules that have received attention over the years are p-essentially normal—including the d-shift Hilbert module, the Hardy and Bergman modules of the unit ball. Arveson conjectured graded submodules over the unit ball inherit this property and provided motivations to seek an affirmative answer. Some positive results have been obtained by Arveson and Douglas. However, the problem has been resistant. In dimensions $d = 2, 3$, this paper shows that the Arveson's conjecture is true. In any dimension, the paper also gives an affirmative answer in the case of the graded principal submodule. Finally, the paper is associated with K-homology invariants arising from graded quotient modules, by which geometry of the quotient modules and geometry of algebraic varieties are connected. In dimensions $d = 2, 3$, it is shown that K-homology invariants determined by graded quotients are nontrivial. The paper also establishes results on p-smoothness of K-homology elements, and gives an explicit expression for K-homology invariant in dimension $d = 2$.

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Contents

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1 Introduction

In the study of multivariable operator theory, there is a natural approach via Hilbert module [21]. Let $T = (T_1, \ldots, T_d)$ be a tuple of commuting operators acting on a Hilbert space H. Then, one naturally makes H into a Hilbert module over the polynomial ring $C[z_1, \ldots, z_d]$. The $C[z_1, \ldots, z_d]$-module structure is define by

$$p \cdot \xi = p(T_1, \ldots, T_d)\xi, \quad p \in C[z_1, \ldots, z_d], \quad \xi \in H.$$

In Arveson's language [5], a Hilbert module is said to be essentially normal if the cross-commutators $T_k^* T_j - T_j T_k^*$ of its ambient operators are all compact, and more specifically, p-essentially normal if the cross-commutators belong to the Schatten class $\mathcal{L}^p$, where $p \in [1, \infty)$. For essentially normal Hilbert modules, Douglas called such Hilbert modules as essentially reductive, see [16–19,21]. The study of essential normality and p-essential normality facilitates the introduction of techniques and methods drawn from algebraic geometry, homology theory and complex analysis, etc, and this study establishes important connections between operator theory and these branches. Essentially all of the basic Hilbert modules that have received attention over the years are p-essentially normal–including the d-shift Hilbert module, the Hardy and Bergman modules of the unit ball. Arveson conjectures that graded submodules over the unit ball inherit this property and seeks an affirmative answer [3–7].

Observing the following example will serve as a good illustration of motivations. Let I be a homogeneous ideal of the polynomial ring $C[z_1, \ldots, z_d]$, and let $[I]$ be the graded submodule of the Bergman module $L_a^2(\mathbb{B}_d)$ generated by I. Then there corresponds a graded quotient submodule $L_a^2(\mathbb{B}_d)/[I]$ (naturally identical with $L_a^2(\mathbb{B}_d) \ominus [I]$) with the module action endowed by d-tuple $\{S_1, \ldots, S_d\}$ obtained by compressing $\{M_{z_1}, \ldots, M_{z_d}\}$ to the quotient. Let $C^*\langle I \rangle$ denote the C^*-algebra on $L_a^2(\mathbb{B}_d)/[I]$ generated by the identity operator and d-tuple $\{S_1, \ldots, S_d\}$. If $[I]$ is essentially normal, then applying a result in [4,16] here shows that the quotient submodule $L_a^2(\mathbb{B}_d)/[I]$ also is, and hence one has a C^*-extension defined by

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I \rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0. \tag{1.1}$$

This extension yields an odd K-homology element e_I for the variety $\mathcal{Z}(I)$ by BDF-theory[9], where $\mathcal{Z}(I)$ denotes $Z(I) \cap \partial\mathbb{B}_d$. As the first step, one would expect this K-homology element e_I to be nontrivial [7], which would establish connections between operator theory, index theory and algebraic geometry. A more challenging problem is which element of the $K_1(\mathcal{Z}(I))$ is obtained. Douglas suspected that this K-homology element in fact is identical with the fundamental class for the variety $\mathcal{Z}(I)$ [17].

Concerning p-essential normality of the graded quotient module $L_a^2(\mathbb{B}_d)/[I]$, it naturally arises from the study of p-smoothness and index theory of K-homology element e_I [20,24]. Furthermore, a possible efficient connection to geometry is the Connes' cyclic cohomology, a generalized de Rham cohomology which is essentially important in the noncommutative geometry [13]. Related to p-summable Fredholm modules in the cyclic cohomology, p-essential normalty will play an role in the calculation of Chern character. This requires a deep understanding on topology and geometry of algebraic varieties. In the case of the bounded strongly pseudo-convex domains, the reader might consult [17,18] for related discussions.

For the study of essential normality and p-essential normality, there are many other motivations coming from topology and algebraic geometry. Considering an analytic manifold M in the projective space $\mathbb{P}^n$, by a theorem of Chow [34], M can be obtained by the zero variety of a homogeneous ideal, with suitable criterion which ensures M smooth. That fact implies that M contains plenty of subvariety. By an elementary result of algebraic geometry, each subvariety of M determines a fundamental cohomology element in $H^*(M)$. The famous Hodge's conjecture asserts, roughly speaking, that whether there are enough subvarieties such that their fundamental cohomology classes rationally generate the cohomology group $H^{(-,-)}(M)$. The study of essentially normal Hilbert modules may provide a connection between operator theory and Hodge's conjecture. For a smooth homogeneous variety $\mathcal{Z}(J)$, considering any homogeneous polynomial ideal $I \supseteq J$ and then $\mathcal{Z}(I) \subseteq \mathcal{Z}(J)$, one wonders whether the fundamental class of $\mathcal{Z}(I)$ in cohomology group $H^*(\mathcal{Z}(J))$ is identical with the image of $e_I \in K_1(\mathcal{Z}(I)) \mapsto K_1(\mathcal{Z}(J))$. Furthermore, one can analogously consider the subgroup of $K_1(\mathcal{Z}(J))$ generated by the images of e_I for all $I \supseteq J$. There is even no guess about the possible result in that Hilbert module analogs and the connection with the Hodge's conjecture.

This paper is devoted to the Arveson's conjecture and K-homology invariant problem above mentioned. Some positive results were obtained by Arveson and Douglas. For the d-shift Hilbert module with finite multiplicity, Arveson established p-essential normality in the case the submodule is generated by monomials [4]. That result on "monomial" submodules was generalized by Douglas to cases in which the d-shift is replaced by more general weighted shifts [16]. Furthermore, Arveson has developed a theory of "Standard Hilbert modules" in an effort to establish his conjecture [5]. Recently, Douglas indicated and discussed a new kind of index theorem arising from the study of essential normality [17].

For convenience of discussion, we will mainly work on the d-shift Hilbert module H_d^2 over the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$. It is worth noticing that the same arguments parallel run through the most "natural" Hilbert modules over the unit ball, especially the Hardy module and the Bergman module.

This paper is organized as follows. Section 2 shows that in any dimension d, each graded principal submodule is p-essentially normal for $p > d$, the proof of which is considerably technical, based on two interesting inequalities of multiplication operators. Therefore, the submodule $[z_1^n + z_2^n + z_3^n]$ of 3-shift Hilbert module on which focused by Arveson [5,7], is p-essentially normal for $p > 3$.

Section 3 is devoted to the Arveson's conjecture in dimension $d = 2, 3$. Using a method on trace estimation, it is shown that in dimension $d = 2$, each graded submodule of $H_d^2 \otimes \mathbb{C}^r$ is p-essentially normal for $p > 2$. Combining trace estimation and techniques from commutative algebra, a more technical analysis shows that in the case of dimension $d = 3$, each graded submodule of H_d^2 is p-essentially normal for $p > 3$.

Concerning the study of p-smoothness of extensions arising from quotient submodules, in scalar-valued version, Douglas [17] raised a refinement for Arveson's conjecture as follows: Let I be an ideal of $C[z_1, \dots, z_d]$, then the quotient module $L_a^2(\mathbb{B}_d)/[I]$ is p-essentially normal for $p > \dim_{\mathbb{C}} Z(I)$. In dimensions $d = 2, 3$, Sect. 4 shows that the assertion is true.

Section 5 is devoted to the study of K-homology invariants arising from graded quotient modules. For each essentially normal graded submodule $M = [I]$, considering the corresponding quotient module $H_d^2/[I]$, and let $C^*\langle I\rangle$ be the C^*-algebra on $H_d^2/[I]$ generated by the identity operator and d-tuple $\{S_1, \dots, S_d\}$ obtained by compressing $\{M_{z_1}, \dots, M_{z_d}\}$ to the quotient module. Then one obtains a C^*-extension as in (1.1). This extension yields an odd K-homology element e_I for the variety $\mathcal{Z}(I)$. It is shown that this element e_I is nontrivial in dimensions $d = 2,\ 3$. The proofs are based on methods of commutative algebra and the characteristic space theory [12,29]. This result applies to the submodule $[z_1^n + z_2^n - z_3^n]$ of 3-shift Hilbert module to obtain a nontrivial geometric invariant for noncommutative Fermat curve "$X^n + Y^n = Z^n$" on which focused by Arveson [5,7]. It is remarkable that the K-homology elements doesn't depend on the kernel Hilbert module with which one begins. In [17], Douglas suspected the same thing is true for more general case, especially in the case of algebraic variety. For general cases, a deep understanding on topology and geometry of algebraic variety is required. Douglas suggested, in a private communication, that one needs a generalization of the calculus of pseudo-differential operators to the context of complex algebraic varieties.

2 Each graded principal submodule is *p*-essentially normal

We will work on the d-shift Hilbert module H_d^2 over the unit ball $\mathbb{B}_d$ of the d-dimensional complex space $\mathbb{C}^d$, where $\mathbb{B}_d = \{z \in \mathbb{C}^d : |z| < 1\}$. Recall H_d^2 is the Hilbert space of analytic function determined by the reproducing kernel $K_w(z) = \frac{1}{1-\langle z,w\rangle}$, where $\langle z, w\rangle = \sum_{i=1}^d z_i \overline{w_i}$. It is easy to verify that H_d^2 has a canonical orthonormal basis $\{(\frac{|\alpha|!}{\alpha!})^{\frac{1}{2}} z^\alpha\}$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ run over multi-indices of nonnegative integers, and $\alpha! = \alpha_1! \dots \alpha_d!$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

The d-shift Hilbert module H_d^2, known as the symmetric Fock space, plays a basic role in the study of multivariable operator theory. The space was first considered by Drury in [23] to generalize Von Neumann's inequality. From the point of view of Hilbert modules, the module has been comprehensively investigated by Arveson, we refer the reader to the references [1–5,7] and [26–28] for a far-reaching operator-theoretic and index-theoretic developments of the Hilbert module. Arveson raised a conjecture about the essentially normality or p-essentially normality ($p > d$) of graded submodules of $H_d^2 \otimes \mathbb{C}^r$ with finite multiplicity r [3–7].

Let M be a submodule of $H_d^2 \otimes \mathbb{C}^r$. Write $R_i = (M_{z_i} \otimes I)|_M$ for the restriction of $M_{z_i} \otimes I$ to M and $S_i = P_{M^\perp}(M_{z_i} \otimes I)|_{M^\perp}$ for the compression of $M_{z_i} \otimes I$ to $M^\perp$

for $i = 1, \ldots, d$. In Arveson's language [5], the submodule M (the quotient module $M^{\perp}$) is said to be p-essentially normal if the cross-commutators $R_i R_j^* - R_j^* R_i$ ($S_i S_j^* - S_j^* S_i$) are in $\mathcal{L}^p$ for $i, j = 1, \ldots, d$, where $p \in [1, \infty]$, and where $\mathcal{L}^\infty$ means the ideal $\mathcal{K}$ of compact operators.

Let $\mathbb{H}_n$ be the space of homogeneous polynomials with degree n. Then $H_d^2 \otimes \mathbb{C}^r$ has a canonical graded decomposition $H_d^2 \otimes \mathbb{C}^r = \oplus_n (\mathbb{H}_n \otimes \mathbb{C}^r)$. A submodule M is called graded if $M = \oplus_n M \cap (\mathbb{H}_n \otimes \mathbb{C}^r)$. Equivalently, M is generated by finitely many vector-valued homogeneous polynomials. When M is generated by one polynomial, we say that M is a principal submodule. In this section we will show that the Arveson's conjecture is true in the case of the graded principal submodule.

We begin with the following lemma that comes from [4,5]. The lemma also appeared in [16].

Lemma 2.1 *Let M be a submodule of $H_d^2 \otimes \mathbb{C}^r$. Then the followings are equivalent: for $p > d$,*

1. *M is p-essentially normal,*
2. *$M^{\perp}$ is p-essentially normal,*
3. *$[P_M, M_{z_i}] = P_M M_{z_i} - M_{z_i} P_M$ are in $\mathcal{L}^{2p}$ for $1 \le i \le d$.*

The following is the main result in this section.

Theorem 2.2 *Each graded principal submodule of H_d^2 is p-essentially normal for $p > d$, that is, if q be a homogeneous polynomial, then the submodule $[q]$ generated by q is p-essentially normal for $p > d$.*

From the above theorem, the submodule $[z_1^n + z_2^n + z_3^n]$ of 3-shift Hilbert module H_3^2 is p-essentially normal for $p > 3$. This answers a problem in [5,7].

The proof of Theorem 2.2 is considerably technical. We begin with some notations. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta = (\beta_1, \ldots, \beta_d)$ be multi-indices of nonnegative integers. As usual, set $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_d \pm \beta_d)$ and $z^\alpha = z_1^{\alpha_1}, \ldots, z_d^{\alpha_d}$, $\partial^\alpha = \partial_1^{\alpha_1}, \ldots, \partial_d^{\alpha_d}$, where $\partial_j = \frac{\partial}{\partial z_j}$. Given two functions g, h, using iteration of the formula $\partial_j(gh) = (\partial_j g)\, h + g\, (\partial_j h)$, one has the following Newton–Leibnitz formula

$$\partial^\alpha (gh) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta g)(\partial^\gamma h). \tag{2.1}$$

Given a polynomial $p(z_1, \ldots, z_d) = \sum_\alpha a_\alpha z^\alpha$, define the differential operator $p(\partial) = \sum_\alpha a_\alpha \partial^\alpha$. In general, $p(\partial)$ acting in H_d^2, is unbounded, but it is densely defined and closed. For a polynomial q, applying the Newton-Leibnitz formula (2.1), we have

$$p(\partial)\, M_q = \sum_\alpha \frac{1}{\alpha!} M_{\partial^\alpha q}\, (\partial^\alpha p)(\partial). \tag{2.2}$$

Define the number operator N acting in $H_d^2 = \bigoplus_n \mathbb{H}_n$ by $Nh = nh$ if $h \in \mathbb{H}_n$. The operator N is unbounded, and its n-th eigenspace is $\mathbb{H}_n$. Given a function $F : \mathbb{Z}_+ \to \mathbb{R}$, the operator $F(N)$ acting in H_d^2 is understood as $F(N)h = F(n)h$ if $h \in \mathbb{H}_n$. Let A be an operator acting in H_d^2, A is called graded if there exists an integer m such that $A\mathbb{H}_n \subseteq \mathbb{H}_{n+m}$ for $n = 0, 1, \ldots$, where the integer m is called the degree of A. If A is graded and deg $A = m$, then A^* also is graded and deg $A^* = -m$. Moreover, we have

$$AF(N) = F(N-m)A. \tag{2.3}$$

Let $p = \sum_\alpha a_\alpha z^\alpha$ be a homogeneous polynomial with the degree m, and write $\dot{p} = \sum_\alpha \bar{a}_\alpha z^\alpha$. We consider multiplication operator M_p acting on H_d^2. Then a straightforward computation shows that

$$\dot{p}(\partial)E_n = \frac{n!}{(n-m)!}M_p^* E_n, \;\; n = 0, 1, \ldots,$$

where E_n denotes the projection from H_d^2 onto the space $\mathbb{H}_n$ of homogeneous polynomials with the degree n. For each nonnegative integer k, we define $F_k : \mathbb{Z}_+ \to \mathbb{R}$ by $F_k(n) = \frac{(n+k)!}{n!}$. Then we have

$$\dot{p}(\partial) = F_m(N)M_p^*. \tag{2.4}$$

Let q be a homogeneous polynomial with the degree l. Combining (2.2), (2.3) and (2.4) we have

$$\begin{aligned} F_m(N)M_p^* M_q &= \dot{p}(\partial)M_q = \sum_\alpha \frac{1}{\alpha!} M_{\partial^\alpha q}(\partial^\alpha \dot{p})(\partial) \\ &= \sum_\alpha \frac{1}{\alpha!} M_{\partial^\alpha q} F_{m-|\alpha|}(N) M_{\partial^\alpha p}^* \\ &= \sum_\alpha \frac{1}{\alpha!} F_{m-|\alpha|}(N-l+|\alpha|) M_{\partial^\alpha q} M_{\partial^\alpha p}^*. \end{aligned}$$

This implies that

$$M_p^* M_q = \sum_\alpha \frac{1}{\alpha!} F_\alpha(N) M_{\partial^\alpha q} M_{\partial^\alpha p}^*, \tag{2.5}$$

where $F_\alpha : \mathbb{Z}_+ \to \mathbb{R}$ is defined by

$$F_\alpha(n) = \frac{F_{m-|\alpha|}(n-l+|\alpha|)}{F_m(n)} = \frac{n!(n+m-l)!}{(n+m)!(n-l+|\alpha|)!}.$$

Let p be a homogeneous polynomial with degree m, applying (2.5) gives that for $j = 1, \ldots, d$,

$$M_{z_j}^* M_p = (N-m+1)(N+1)^{-1} M_p M_{z_j}^* + (N+1)^{-1} M_{\partial_j p}. \tag{2.6}$$

When $p = q$, the equality (2.5) gives that

$$M_p^* M_p = \sum_\alpha \frac{1}{\alpha!} F_\alpha(N) M_{\partial^\alpha p} M_{\partial^\alpha p}^*, \tag{2.7}$$

where the operator $F_\alpha(N)$ is defined by the function

$$F_\alpha(n) = \frac{F_{m-|\alpha|}(n-m+|\alpha|)}{F_m(n)} = \frac{n!^2}{(n+m)!(n-m+|\alpha|)!}.$$

The following inequalities will play a basic role in the proof of Theorem 2.2.

Proposition 2.3 *Let p be a homogeneous polynomial with degree m. Then there exist positive constants C_1 and C_2 such that*

1. $M_p^* M_p \geq C_1 (N+m)^{-1} \sum_{j=1}^d M_{\partial_j p}^* M_{\partial_j p}$;
2. $M_p^* M_p \geq C_2 (N+1)^{-1} \ldots (N+m)^{-1}$.

Proof Let e_j denote the multi-index with 1 in the j-th position and 0 for all others. By (2.7), we have

$$M_{\partial_j p}^* M_{\partial_j p} = \sum_\beta \frac{1}{\beta!} G_\beta(N) M_{\partial^{\beta+e_j} p} M_{\partial^{\beta+e_j} p}^* \tag{2.8}$$

where the operator $G_\beta(N)$ is defined by the function $G_\beta(n) = \frac{F_{m-1-|\beta|}(n-m+|\beta|+1)}{F_{m-1}(n)}$. Setting $\alpha = \beta + e_j$, a simple computation gives that $G_\beta(N) = (N+m) F_\alpha(N)$, where $F_\alpha(N)$ is that in (2.7). Therefore, from (2.8) and (2.7), there exists a positive constant C_0 such that

$$\begin{aligned} M_{\partial_j p}^* M_{\partial_j p} &= (N+m) \sum_{\alpha:\alpha_j \geq 1} \frac{\alpha_j}{\alpha!} F_\alpha(N) M_{\partial^\alpha p} M_{\partial^\alpha p}^* \\ &\leq C_0 (N+m) \sum_\alpha \frac{1}{\alpha!} F_\alpha(N) M_{\partial^\alpha p} M_{\partial^\alpha p}^* \\ &= C_0 (N+m) M_p^* M_p. \end{aligned}$$

The inequality (1) comes easily from the above. The inequality (2) is an immediate consequence of the inequality (1), completing the proof. □

Let p be a homogeneous polynomial with deg $p = m$. The submodule $[p]$, generated by p, is the closure of the algebraic submodule $\{ph : h \in C[z_1, \ldots, z_d]\}$. Define a densely defined operator S on H_d^2 as follows

$$S[p]^\perp = 0, \quad S(ph) = (N+m)^{-\frac{m}{2}} h, \ \ h \in C[z_1, \ldots, z_d]. \tag{2.9}$$

Lemma 2.4 *Both the densely defined operators S and $(N+1)^{\frac{m-1}{2}} M_{\partial_j p} S$ are bounded, and hence they can be extended as the bounded operators on H_d^2.*

Proof It is enough to show that the operators are bounded on the subspace $\{ph : h \in C[z_1, \dots, z_d]\}$. For any polynomial h, applying Proposition 2.3 (2) shows that there is a positive constant C_2 such that

$$\|(N+m)^{-\frac{m}{2}} h\|^2 = \langle (N+m)^{-m} h, h\rangle \le C_2 \langle M_p^* M_p h, h\rangle = C_2 \|p h\|^2.$$

By the definition (2.9) of S, we see that S is bounded.

Using the formula (2.3) and Proposition 2.3(1), there exists a positive constant C_1 such that

$$\begin{aligned}
&\|(N+1)^{\frac{m-1}{2}} M_{\partial_j p} S(ph)\|^2 = \|(N+1)^{\frac{m-1}{2}} M_{\partial_j p} (N+m)^{-\frac{m}{2}} h\|^2 \\
&= \|(N+1)^{-\frac{1}{2}} M_{\partial_j p} h\|^2 = \langle (N+m)^{-1} M_{\partial_j p}^* M_{\partial_j p} h, h\rangle \\
&\le C_1 \langle M_p^* M_p h, h\rangle = C_1 \|ph\|^2.
\end{aligned}$$

Therefore, the operator $(N+1)^{\frac{m-1}{2}} M_{\partial_j p} S$ is bounded, completing the proof. □

Let Q_p be the orthogonal projection from H_d^2 onto $[p]$. It is easy to check

$$Q_p = N^{\frac{m}{2}} M_p S. \tag{2.10}$$

The proof of Theorem 2.2 By Lemma 2.1 (3), it is enough to show that $[Q_p, M_{z_j}] \in \mathcal{L}^{2s}$ for $s > d$ and $1 \le j \le d$. Setting $Q_p^\perp = I - Q_p$, and applying (2.3), (2.6) and (2.10) we have

$$\begin{aligned}
&M_{z_j}^* Q_p - Q_p M_{z_j}^* = M_{z_j}^* Q_p - Q_p M_{z_j}^* Q_p = Q_p^\perp M_{z_j}^* Q_p \\
&\overset{(2.10)}{=} Q_p^\perp M_{z_j}^* N^{\frac{m}{2}} M_p S \\
&\overset{(2.3)}{=} (N+1)^{\frac{m}{2}} Q_p^\perp M_{z_j}^* M_p S \\
&\overset{(2.6)}{=} (N+1)^{\frac{m}{2}} Q_p^\perp \Big((N-m+1)(N+1)^{-1} M_p M_{z_j}^* + (N+1)^{-1} M_{\partial_j p} \Big) S \\
&= (N+1)^{\frac{m}{2}} (N-m+1)(N+1)^{-1} Q_p^\perp M_p M_{z_j}^* S + (N+1)^{\frac{m}{2}-1} Q_p^\perp M_{\partial_j p} S \\
&= (N+1)^{-\frac{1}{2}} Q_p^\perp (N+1)^{\frac{m-1}{2}} M_{\partial_j p} S,
\end{aligned}$$

where the last equation follows from $Q_p^\perp M_p = 0$.

Setting $A_j = Q_p^\perp (N+1)^{\frac{m-1}{2}} M_{\partial_j p} S$, Lemma 2.4 shows that A_j is a bounded graded operator with $\deg A_j = -1$. The above reasoning gives the equality $[Q_p, M_{z_j}] = A_j^* (N+1)^{-\frac{1}{2}}$. Since $(N+1)^{-\frac{1}{2}} \in \mathcal{L}^{2s}$ for $s > d$, we see that $[Q_p, M_{z_j}] \in \mathcal{L}^{2s}$ for $s > d$ and $1 \le j \le d$, completing the proof. □

As a consequence of the above proof, the following corollary will be used in the next section.

Corollary 2.5 *There exists a bounded graded operator B_j with* $\deg B_j = 1$ *such that*

$$C_j = Q_p M_{z_j} Q_p^{\perp} = [Q_p, M_{z_j}] = B_j\,(N+1)^{-\frac{1}{2}}, \quad j = 1, \ldots, d.$$

In the essentially same way as done in the scalar-valued case, Theorem 2.2 can be generalized to the vector-valued version, and described as follows. Let $\mathbf{p} = (p_1, \ldots, p_r) \in C[z_1, \ldots, z_d] \otimes \mathbb{C}^r$, where each component p_i is a homogeneous polynomial, and not necessarily have the same degree. Considering the submodule $[\mathbf{p}]$ of $H_d^2 \otimes \mathbb{C}^r$ generated by $\mathbf{p}$, then $[\mathbf{p}]$ is the closure of the set $\{\mathbf{p}h : h \in C[z_1, \ldots, z_d]\}$. Write $m_1 = \deg p_1, \ldots, m_r = \deg p_r$, and set $m = \max\{m_1, \ldots, m_r\}$. We define the operator $S : H_d^2 \otimes \mathbb{C}^r \to H_d^2$ by

$$S\,[\mathbf{p}]^{\perp} = 0, \quad S\,(\mathbf{p}h) = (N+m)^{-\frac{m}{2}} h,\;\; h \in C[z_1, \ldots, z_d].$$

Then the operator S is bounded. Let $M_{\mathbf{p}} : H_d^2 \to H_d^2 \otimes \mathbb{C}^r$ be a multiplication operator defined by $M_{\mathbf{p}} h = \mathbf{p}h$ for $h \in H_d^2$, and let $Q_{\mathbf{p}}$ denote the orthogonal projection from $H_d^2 \otimes \mathbb{C}^r$ onto $[\mathbf{p}]$, then

$$Q_{\mathbf{p}} = M_{\mathbf{p}} (N+m)^{\frac{m}{2}} S.$$

By a slight careful modification of the proof of Theorem 2.2, one can obtain that $Q_{\mathbf{p}}^{\perp} M_{z_j}^* Q_{\mathbf{p}} \in \mathcal{L}^{2p}$ for $p > d$. Therefore, by Lemma 2.1(3), the principal submodule $[\mathbf{p}]$ is p-essentially normal for $p > d$.

3 In dimensions $d = 2,\ 3$, each graded submodule is p-essentially normal

The following theorem shows that Arveson's conjecture is true in dimension $d = 2$. Essentially, this is a generalization of [31, Theorem 2.4].

Theorem 3.1 *If $d = 2$ and M is a graded submodule of $H_d^2 \otimes \mathbb{C}^r$, then M is p-essentially normal for $p > 2$.*

Proof Since M is graded, this means that it can be decomposed as

$$M = M_0 \oplus M_1 \oplus \cdots \oplus M_n \oplus \cdots,$$

where $M_n \subseteq \mathbb{H}_n \otimes \mathbb{C}^r$ for $n = 0, 1, \ldots$. Writing A_i for $[P_M, M_{z_i}][P_M, M_{z_i}]^*$ for $i = 1, 2$, a simple verifying gives that

$$A_i = P_M M_{z_i} M_{z_i}^* P_M - M_{z_i} P_M M_{z_i}^*,$$

and hence the operator A_i maps M_n to M_n. Thus we have

$$\begin{aligned} \mathrm{Trace}(A_i|_{M_n}) &= \mathrm{Trace}(P_{M_n} M_{z_i} M_{z_i}^* P_{M_n}) - \mathrm{Trace}(P_{M_n} M_{z_i} P_{M_{n-1}} M_{z_i}^* P_{M_n}) \\ &= \mathrm{Trace}(P_{M_n} M_{z_i} M_{z_i}^* P_{M_n}) - \mathrm{Trace}(P_{M_{n-1}} M_{z_i}^* M_{z_i} P_{M_{n-1}}). \end{aligned}$$

By [1, Lemma 2.8 and Propostion 5.3], it holds that

$$M_{z_1}M_{z_1}^* + M_{z_2}M_{z_2}^* = I - 1 \otimes 1, \quad M_{z_1}^*M_{z_1} + M_{z_2}^*M_{z_2} = \frac{N+2}{N+1}.$$

Hence

$$\mathrm{Trace}(A_1 + A_2)|_{M_n} = \dim\ M_n - \frac{n+1}{n}\dim\ M_{n-1}.$$

Applying [2, Theorem 4.2], there exist integers a, b and N such that $\dim\ M_n = a+bn$ for all $n \geq N$. Therefore, when $n \geq N+1$, we have

$$\mathrm{Trace}(A_1 + A_2)|_{M_n} = \dim\ M_n - \frac{n+1}{n}\dim\ M_{n-1} = \frac{b-a}{n}.$$

From Sect. IV.2 in [13], the operator $A_1 + A_2$ belongs to the weak trace class $\mathcal{L}^{(1,\infty)}$, and hence $A_1 + A_2 \in \mathcal{L}^t$ for $t > 1$. This implies that all $[P_M, M_{z_i}]$ belong to Schatten class $\mathcal{L}^q$ for $q > 2$. The desired conclusion comes from Lemma 2.1, completing the proof. □

In what follows we will show Arveson's conjecture is true in the case of the dimension $d = 3$.

Theorem 3.2 *In dimension $d = 3$, each graded submodule of H_d^2 is p-essentially normal for $p > 3$.*

In order to prove this theorem we require some notations and lemmas.

The Hilbert polynomial is an invariant associated with the graded modules. We refer the reader to references [2,25,38] for more details. For convenience, we remark a variation of [38, Vol(II), Chapter VII, Th.41 and Th.42'] as follows.

Lemma 3.3 *Let L be a graded algebraic submodule of $C[z_1, \dots, z_d] \otimes \mathbb{C}^r$ over the polynomial ring $C[z_1, \dots, z_d]$, and let*

$$L = \sum_n L_n, \quad C[z_1, \dots, z_d] \otimes \mathbb{C}^r = \sum_n \mathbb{H}_n \otimes \mathbb{C}^r$$

be graded decompositions of L and $C[z_1, \dots, z_d] \otimes \mathbb{C}^r$, respectively. Then for sufficiently large n, the dimension $\dim\ \mathbb{H}_n \otimes \mathbb{C}^r / L_n$ *is a polynomial whose degree is equal to the projective dimension of the zero variety determined by the annihilator ideal* $\mathrm{Ann}(C[z_1, \dots, z_d] \otimes \mathbb{C}^r / M)$.

Let M be a graded submodule of H_d^2, then there exists a unique homogeneous ideal I of $C[z_1, \dots, z_d]$ such that $M = [I]$, where $[I]$ denotes the closure of I in H_d^2. In fact, $I = M \cap C[z_1, \dots, z_d]$. For each homogeneous ideal I, the ideal I can be uniquely written as $I = pL$, which is called the Beurling form of I, where p is the greatest common divisor of I, i.e., $p = \mathrm{GCD}(I) = \mathrm{GCD}\{q : q \in I\}$. It is easy to check that both p and L are homogeneous. In dimension $d = 3$, by the elementary fact of algebraic geometry, one sees that $\dim_{\mathbb{C}} Z(L) \leq 1$. Let $L = \oplus_n L_n$ be the homogeneous decomposition of L. Applying lemma 3.3 shows that there exists a positive integer N_0 such that when $n \geq N_0$, $\dim \mathbb{H}_n/L_n = l$ for some constant l.

Decompose the coordinate multiplication operators $M_{z_j} (j = 1, 2, 3)$ with respect to $[p] \oplus [p]^{\perp}$ as follows

$$M_{z_j} = \begin{bmatrix} R_j & C_j \\ 0 & S_j \end{bmatrix}.$$

Then we have

$$M_{z_j}^* M_{z_j} = \begin{bmatrix} R_j^* R_j & R_j^* C_j \\ C_j^* R_j & C_j^* C_j + S_j^* S_j \end{bmatrix}, \quad M_{z_j} M_{z_j}^* = \begin{bmatrix} R_j R_j^* + C_j C_j^* & C_j S_j^* \\ S_j C_j^* & S_j S_j^* \end{bmatrix}$$

This gives that

$$R_j^* R_j = Q_p M_{z_j}^* M_{z_j} Q_p, \quad R_j R_j^* = Q_p M_{z_j} M_{z_j}^* Q_p - C_j C_j^*, \tag{3.1}$$

where Q_p is the orthogonal projection from H_d^2 onto $[p]$.

The proof of Theorem 3.2. Let Q_I denote the orthogonal projection from H_d^2 onto $[I]$. From Lemma 2.1(3), it suffices to show that $[Q_I, M_{z_j}] \in \mathcal{L}^{2s}$ for $s > 3$, and $j = 1, 2, 3$. Considering the equality

$$\begin{aligned}[Q_I, M_{z_j}] &= [Q_I, M_{z_j}]Q_p + [Q_I, M_{z_j}]Q_p^{\perp} = [Q_I, M_{z_j}]Q_p + Q_I M_{z_j} Q_p^{\perp} \\ &= [Q_I, M_{z_j}]Q_p + Q_I Q_p M_{z_j} Q_p^{\perp} = [Q_I, M_{z_j}]Q_p + Q_I C_j,\end{aligned}$$

Corollary 2.5 shows $C_j \in \mathcal{L}^{2s}$ for $s > 3$. It remains to show $[Q_I, M_{z_j}]Q_p \in \mathcal{L}^{2s}$ for $s > 3$. Write $Q = Q_p - Q_I$, that is, Q is the orthogonal projection from H_d^2 onto $[p] \ominus [pL]$. By the equalities

$$[Q_I, M_{z_j}]Q_p = -[Q, R_j], \quad j = 1, 2, 3, \tag{3.2}$$

it suffices to show that the following assertion is true

$$\sum_{j=1}^{3} [Q, R_j]^*[Q, R_j] \in \mathcal{L}^s$$

for $s > 3$. Since $QR_j^* Q = R_j^* Q, \quad QR_j Q = QR_j$, we have

$$[Q, R_j]^*[Q, R_j] = QR_j^* R_j Q - R_j^* Q R_j. \tag{3.3}$$

Let $L=\sum_n L_n$, $C[z_1,z_2,z_3]=\sum_n \mathbb{H}_n$ be homogeneous decompositions of L and $C[z_1,z_2,z_3]$, respectively. Then $[p]=\bigoplus_n p\mathbb{H}_n$, $[pL]=\bigoplus_n pL_n$ are homogeneous decompositions of $[p]$ and $[pL]$, respectively, and hence

$$[p]\ominus[pL]=\bigoplus_n (p\mathbb{H}_n\ominus pL_n).$$

Let P_n and Q_n be the orthogonal projections from H_d^2 onto $p\mathbb{H}_n$ and $p\mathbb{H}_n\ominus pL_n$, respectively. Setting $A_n=P_n\left(\sum_{j=1}^3[Q,R_j]^*[Q,R_j]\right)P_n$, then

$$\sum_{j=1}^3[Q,R_j]^*[Q,R_j]=\sum_n A_n.$$

Noticing the fact $\sum_{i=1}^d M_{z_i}M_{z_i}^*=I-1\otimes 1$, $\sum_{i=1}^d M_{z_i}^*M_{z_i}=\frac{N+d}{N+1}$ and applying (3.1), (3.3),

$$\begin{aligned}
A_n&=P_n\left(\sum_{j=1}^3[Q,R_j]^*[Q,R_j]\right)P_n=\sum_{j=1}^3 Q_n[Q,R_j]^*[Q,R_j]Q_n\\
&=Q_n\left(\sum_{j=1}^3 M_{z_j}^*M_{z_j}\right)Q_n-\sum_{j=1}^3 Q_nR_j^*QR_jQ_n\\
&=\frac{m+n+d}{m+n+1}Q_n-\sum_{j=1}^3 Q_nR_j^*Q_{n+1}R_jQ_n.
\end{aligned}$$

Combining Corollary 2.5 and (3.1), we see

$$\begin{aligned}
\text{Trace } A_n&=\frac{m+n+d}{m+n+1}\dim Q_n-\text{Trace}\left(\sum_{j=1}^3 Q_nR_j^*Q_{n+1}R_jQ_n\right)\\
&=\frac{m+n+d}{m+n+1}\dim Q_n-\text{Trace } Q_{n+1}\left(\sum_{j=1}^3 R_jR_j^*\right)Q_{n+1}\\
&=\left(\frac{m+n+d}{m+n+1}\dim Q_n-\dim Q_{n+1}\right)+\text{Trace } Q_{n+1}\left(\sum_{j=1}^3 C_jC_j^*\right)Q_{n+1}\\
&=\left(\frac{m+n+d}{m+n+1}\dim Q_n-\dim Q_{n+1}\right)+\frac{\text{Trace } Q_{n+1}\left(\sum_{j=1}^3 B_jB_j^*\right)Q_{n+1}}{m+n+1}.
\end{aligned}$$

By Lemma 3.3 there exists a natural number N_0 such that when $n \geq N_0$, dim $Q_n = \dim(p\mathbb{H}_n \ominus pL_n) = l$ for some constant l. Therefore,

$$\text{Trace } Q_{n+1} \left(\sum_{j=1}^{3} B_j B_j^* \right) Q_{n+1} \leq C_0 \text{ Trace } Q_{n+1} = C_0\, l,$$

where $C_0 = \| \sum_{j=1}^{3} B_j B_j^* \|$. This means that there exists a positive constant C such that when $n \geq N_0$,

$$\text{Trace } A_n \leq \frac{C}{m+n+1}. \tag{3.4}$$

Therefore, $A_n \in \mathcal{L}^{(1,\infty)}$ and it follows that

$$\sum_n A_n = \sum_{j=1}^{3} [Q, R_j]^* [Q, R_j] \in \mathcal{L}^s$$

for $s > 1$. This gives the desired assertion, completing the proof. □

Using the proof of Theorem 3.3, we have

Proposition 3.4 *Assume that I is a homogeneous ideal of $C[z_1, \ldots, z_d]$ and $I = pL$ is its Beurling form. If the complex dimension of the zero variety $Z(L)$ $\dim_{\mathbb{C}} Z(L) = 0$, or 1, then $[I]$ is p-essentially normal for $p > d$.*

4 *p*-essentially normal graded quotient modules

To any submodule M of $H_d^2 \otimes \mathbb{C}^r$, there corresponds a quotient submodule $H_d^2 \otimes \mathbb{C}^r / M$. From Lemma 2.1, if the submodule is p-essentially normal, then the corresponding quotient submodule necessarily enjoys this property. However, for the most "natural" examples, the quotient submodules enjoy more strong property than this. To study p-smoothness of extensions arising from quotient submodules, Douglas [17] raised a refinement for Arveson's conjecture as follows:

Douglas's conjecture Let I be an ideal of $C[z_1, \ldots, z_d]$. Then the quotient module $L_a^2(\mathbb{B}_d)/[I]$ is p-essentially normal for $p > \dim_{\mathbb{C}} Z(I)$.

For convenience of discussion, we will work on the d-shift Hilbert module H_d^2. The same arguments parallel run through the most "natural" Hilbert modules over the unit ball, especially the Bergman module over the unit ball. It is shown that in dimension $d = 2, 3$, if I is homogeneous, then the quotient module $H_d^2/[I]$ is p-essentially normal for $p > \dim_{\mathbb{C}} Z(I)$.

The following result comes from the discussion with Douglas.

Proposition 4.1 *Let M be a graded submodule of $H_d^2 \otimes \mathbb{C}^r$. If the zero variety of the annihilator ideal* $\mathrm{Ann}(H_d^2 \otimes \mathbb{C}^r/M)$ *has the complex dimension* ≤ 1, *Then the quotient module $H_d^2 \otimes \mathbb{C}^r/M$ is p-essentially normal for $p > 1$.*

Proof Let $\{S_1, \ldots, S_d\}$ be the tuple obtained by compressing $\{M_{z_1}, \ldots, M_{z_d}\}$ to $M^\perp$ (here $H_d^2 \otimes \mathbb{C}^r/M$ is naturally identified with $M^\perp$). Writing M_i for M_{z_i} for $i = 1, \ldots, d$, then it is easy to check that

$$[S_i^*, S_j] = P^\perp[M_i^*, M_j]P^\perp - [P^\perp, M_i]^*[P^\perp, M_j], \tag{4.1}$$

where $P^\perp$ denotes the projection onto $M^\perp$. From [1, Proposition 5.3]

$$\begin{aligned} M_i^* M_j - M_j M_i^* &= -(N+1)^{-1} M_j M_i^*, \quad \text{if } i \neq j, \quad \text{and} \\ M_i^* M_i - M_i M_i^* &= (N+1)^{-1} - (N+1)^{-1} M_i M_i^*. \end{aligned} \tag{4.2}$$

Since the zero variety of the annihilator ideal has the complex dimension ≤ 1, applying Lemma 3.3 shows that there exists some positive integer K_0 such that when $n > K_0$, the dimension $\dim P^\perp E_n P^\perp$ is a constant, here E_n is the orthogonal projection from $H_d^2 \otimes \mathbb{C}^r$ onto $\mathbb{H}_n \otimes \mathbb{C}^r$. Therefore, for $p > 1$, the operator $P^\perp(N+1)^{-1}P^\perp \in \mathcal{L}^p$ and it follows from (4.2) that

$$P^\perp[M_i^*, M_j]P^\perp \in \mathcal{L}^p. \tag{4.3}$$

Setting $B_i = [P^\perp, M_i]^*[P^\perp, M_i]$, $i = 1, \ldots, d$, then $B_i \geq 0$ and a simple verifying shows that

$$B_i = P^\perp M_i^* M_i P^\perp - M_i^* P^\perp M_i P^\perp.$$

Let $M^\perp = N_0 \oplus N_1 \oplus \cdots \oplus N_n \oplus \cdots$ be the homogeneous decomposition of $M^\perp$, and $P_n^\perp$ denote the orthogonal projection onto N_n. Put

$$B = B_1 + \cdots + B_d, \quad B^{(n)} = P_n^\perp B P_n^\perp.$$

Then as done in the proof of Theorem 3.2, when $n > K_0$, there exists a positive constant C such that

$$\mathrm{Trace}\, B^{(n)} \leq \frac{C}{n+1}.$$

Therefore, $B \in \mathcal{L}^{(1,\infty)} \subset \mathcal{L}^t$ for $t > 1$, and it follows that

$$[P^\perp, M_i] \in \mathcal{L}^t$$

for $t > 2$ and $i = 1, \ldots, d$. Applying this fact and (4.3) to (4.1) shows that $[S_i^*, S_j] \in \mathcal{L}^p$ for $p > 1$, that is, the quotient module is p-essentially normal for $p > 1$. □

The following results support Douglas's conjecture.

Proposition 4.2 *In dimensions $d = 2, 3$, if I is homogeneous, then the quotient module $H_d^2/[I]$ is p-essentially normal for $p > \dim_{\mathbb{C}} Z(I)$.*

The proof of Proposition 4.2 In dimension $d = 2$, we apply Proposition 4.1 to obtain the desired conclusion. It remains to prove the case of dimension $d = 3$. Since $I \neq 0$, this means $\dim_{\mathbb{C}} Z(I) \leq 2$. When $\dim_{\mathbb{C}} Z(I) \leq 1$, the result comes from Proposition 4.1. In the case $\dim_{\mathbb{C}} Z(I) = 2$, then necessarily, the ideal I has the nonconstant greatest common divisor p, and let $I = pL$ be its Beurling form. Applying Lemma 3.3 shows that there exist some positive integers K_0, a, b, such that when $n > K_0$, the dimension $\dim Q_I^{\perp} E_n Q_I^{\perp} = a + bn$, where $Q_I^{\perp} = I - Q_I$, and Q_I denotes the orthogonal projection from H_d^2 onto $[I]$, E_n is the orthogonal projection from H_d^2 onto $\mathbb{H}_n$. Therefore, for $p > 2$ the operator $Q_I^{\perp}(N+1)^{-1}Q_I^{\perp} \in \mathcal{L}^p$. From (4.2),

$$Q_I^{\perp}[M_i^*, M_j]Q_I^{\perp} \in \mathcal{L}^p, \quad p > 2.$$

To complete the proof, it suffices to prove the following claim:

$$[Q_I, M_j]^*[Q_I, M_j] \in \mathcal{L}^p \quad \text{for } p > 2 \quad \text{and} \quad j = 1, 2, 3. \tag{4.4}$$

Below, We use the same notations as the proof of Theorem 3.2 in Sect. 3. A simple calculation gives

$$\begin{aligned}[Q_I, M_j]^*[Q_I, M_j] = [Q, R_j]^*[Q, R_j] - [Q, R_j]^* Q_I C_j \\ - C_j^* Q_I [Q, R_j] + C_j^* Q_I C_j.\end{aligned}$$

From the proof of Theorem 3.2, the commutators $[Q, R_j] \in \mathcal{L}^p$ for $p > 2$ and $j = 1, 2, 3$. Therefore it is enough to show that $C_j^* Q_I C_j \in \mathcal{L}^p$ for $p > 2$ and $j = 1, 2, 3$. By Corollary 2.5,

$$\begin{aligned} C_j^* Q_I C_j &= B_j^* Q_I B_j (N+1)^{-\frac{1}{2}}(N+2)^{-\frac{1}{2}} \\ &= B_j^* Q_I B_j (N+1)^{-\frac{1}{2}}(N+2)^{-\frac{1}{2}} Q_p^{\perp} \\ &= \frac{(N+1)^{\frac{1}{2}}}{(N+2)^{\frac{1}{2}}} B_j^* Q_I B_j Q_p^{\perp}(N+1)^{-1} Q_p^{\perp}. \end{aligned}$$

It is easy to verify $Q_p^{\perp}(N+1)^{-1}Q_p^{\perp} = Q_p^{\perp}Q_I^{\perp}(N+1)^{-1}Q_I^{\perp}Q_p^{\perp} \in \mathcal{L}^p$ for $p > 2$. The above reasoning shows that the claim (4.4) is true, and hence the proof is complete. □

Applying the same proof of Proposition 4.2, we have

Proposition 4.3 *Let $I = pL$ be the Beurling form of a homogeneous ideal I of $C[z_1, \ldots, z_d]$. If the complex dimension of the zero variety $Z(L)$, $\dim_{\mathbb{C}} Z(L) = 0$, or 1, then the quotient module $H_d^2/[I]$ is p-essentially normal for $p > d - 1$.*

5 K-homology

To study K-homology yielded by graded quotient modules, one naturally is associated with Taylor spectrum and Taylor essential spectrum for operator tuples. There is a large literature concerning Taylor spectrum and Taylor essential spectrum for operator tuples on a Hilbert space. We have made no attempt to compile a comprehensive list of references, but refer the reader's attention to [14,15,35,36]. The spectrum and essential spectrum of graded submodules and quotient modules is well known for experts. Because of the lack of references, we remark a short proof as follows.

Let M be a submodule of H_d^2, and write R_i, S_i for the restriction of M_{z_i} to M, and the compression of M_{z_i} to $M^\perp$ for $i = 1, \dots, d$, respectively.

Theorem 5.1 *Assume that I is a homogeneous ideal of $C[z_1, \dots, z_d]$. For the graded submodule $M = [I]$ of H_d^2, if M is essentially normal, then we have*

1. $\sigma(R_1, \dots, R_d) = \overline{\mathbb{B}_d}$,
2. $\sigma_e(R_1, \dots, R_d) = \partial\mathbb{B}_d$,
3. $\sigma(S_1, \dots, S_d) = Z(I) \cap \overline{\mathbb{B}_d}$,
4. $\sigma_e(S_1, \dots, S_d) = Z(I) \cap \partial\mathbb{B}_d$.

Proof It is easy to see that (1) and (2) come from [33, Sect. 3]. Applying [33] again, it holds that $Z(I) \cap \mathbb{B}_d \subseteq \sigma(S_1, \dots, S_d) \subseteq Z(I) \cap \overline{\mathbb{B}}_d$, and $\sigma_e(S_1, \dots, S_d) \subseteq Z(I) \cap \partial\mathbb{B}_d$. Therefore, it suffices to show that the following inclusion is true

$$\sigma_e(S_1, \dots, S_d) \supseteq Z(I) \cap \partial\mathbb{B}_d.$$

Given any $w = (w_1, \dots, w_d) \in Z(I) \cap \partial\mathbb{B}_d$, we claim that $\sum_{i=1}^d (S_i - w_i)(S_i - w_i)^*$ is not Fredholm. Otherwise, there is a positive finite rank operator F such that the operator

$$A = \sum_{i=1}^d (S_i - w_i)(S_i - w_i)^* + F$$

is positive and invertible. But, by the fact that I is homogeneous, one sees that $tw \in Z(I) \cap \mathbb{B}_d$ and the reproducing kernel $K_{tw} \in M^\perp$ for any $t < 1$. Setting $k_{tw} = \frac{K_{tw}}{\|K_{tw}\|}$, then

$$\begin{aligned}\left\langle \sum_{i=1}^d (S_i - w_i)(S_i - w_i)^* k_{tw},\, k_{tw} \right\rangle &= \left\langle \sum_{i=1}^d (M_{z_i} - w_i)^* k_{tw},\, (M_{z_i} - w_i)^* k_{tw} \right\rangle \\ &= |t-1|^2 |w|^2 \to 0\end{aligned}$$

as $t \to 1$. Since k_{tw} weakly converges to 0 as $t \to 1$, we have that $\|F k_{tw}\| \to 0$. Hence,

$$\langle A k_{tw},\, k_{tw} \rangle \to 0.$$

This leads to a contradiction since the operator A is positive and invertible. This shows that the operator $\sum_{i=1}^{d}(S_i - w_i)(S_i - w_i)^*$ is not Fredholm, and it follows that $\sigma_e(S_1, \dots, S_d) = Z(I) \cap \partial\mathbb{B}_d$, completing the proof. □

Let I be a homogeneous ideal of $C[z_1, \dots, z_d]$. For the graded submodule $M = [I]$ of H_d^2, we use $C^*[I]$ to denote the Toeplitz algebra on M, which is a C^*-algebra generated by $\{I, R_1, \dots, R_d\}$. An easy argument shows that $C^*[I]$ is irreducible. If M is essentially normal, then $C^*[I]$ contains all compact operators $\mathcal{K}$. By Theorem 5.1(2), we have a C^*-extension defined by $C^*[I]$,

$$0 \to \mathcal{K} \hookrightarrow C^*[I] \xrightarrow{\pi} C(\partial\mathbb{B}_d) \to 0,$$

where π is the unital $*$-homomorphism given by $\pi(R_i) = z_i$ for $i = 1, \dots, d$. By [9], this extension yields a canonical element in the odd K-homology group $K_1(\partial\mathbb{B}_d) \cong \mathbb{Z}$. In fact, this K-homology element is identified with the index of the Koszul complex defined by the tuple $\{R_1, \dots, R_d\}$ [8,9,33], and hence it corresponds to $(-1)^d$.

This section will be devoted to K-homology invariants arising from graded quotient modules, by which geometry of the quotient modules and geometry of algebraic varieties are connected. In dimensions $d = 2,\ 3$, it is shown that K-homology invariants determined by graded quotients are nontrivial.

Let I be a homogeneous ideal of $C[z_1, \dots, z_d]$. For the graded submodule $M = [I]$ of H_d^2, we consider the corresponding quotient module H_d^2/M, and naturally identify the quotient module H_d^2/M with $M^\perp$. Let $C^*\langle I\rangle$ denote the C^*-algebra on the graded quotient $[I]^\perp$ generated by Id, $S_1, \dots, S_d$. If $[I]$ is essentially normal, then applying Lemma 2.1 shows that $[I]^\perp$ also is. In this case, using the similar argument as in [5, Proposition 2.5] shows that $C^*\langle I\rangle$ is irreducible and hence it contains all compact operators. Indeed, suppose that there exists a projection P such that for any polynomial q, $PS_q = S_qP$. Since M is graded, $1 \in M^\perp$ and we put $e = P1$. For any $f \in M^\perp$, $f - f(0) \in M^\perp$. Taking a polynomial sequence $\{q_n\}$ with $q_n(0) = 0$ such that q_n converges to $f - f(0)$ in the norm of H_d^2, then we have

$$\begin{aligned}\langle e, f - f(0)\rangle &= \lim_n \langle e, q_n\rangle = \lim_n \langle P1, S_{q_n}1\rangle \\ &= \lim_n \langle 1, S_{q_n}e\rangle = \lim_n \langle 1, q_n e\rangle \\ &= 0.\end{aligned}$$

Taking $f = e$ shows that $\|e - e(0)\|^2 = 0$ and hence $e = e(0)$. Since $e = P1$, it follows that $e(0)^2 = e(0)$. This implies that $e = 0$ or $e = 1$. In the case $e = 0$, for any $f \in M^\perp$, taking a polynomial sequence $\{q_n\}$ such that q_n converges to f, then $S_{q_n}1 = P_{M^\perp}q_n \to f$, and hence

$$PS_{q_n}1 = S_{q_n}P1 = S_{q_n}e = 0 \to Pf.$$

This gives $P = 0$. In the case $e = 1$, the same reasoning shows $P = Id_{M^\perp}$. Therefore, $C^*\langle I\rangle$ is irreducible.

Setting $\mathcal{Z}(I) = Z(I) \cap \partial\mathbb{B}_d$, then from Theorem 5.1(4) we have a C^*-extension defined by $C^*\langle I\rangle$ as follows

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I\rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0. \tag{5.1}$$

This extension yields an odd K-homology element e_I for the variety $\mathcal{Z}(I)$ by BDF-theory[9]. Moreover, in dimension $d = 2, 3$, Proposition 4.2 shows that this element e_I is p-smooth for $p > \dim_{\mathbb{C}} Z(I)$. One naturally would expect this element e_I to be nontrivial [7], which would establish a connection between operator theory and algebraic geometry. A more challenging problem is which element of the $K_1(\mathcal{Z}(I))$ is obtained. Douglas suspected that this K-homology element in fact is identical with the fundamental class for the variety $\mathcal{Z}(I)$ [17].

In what follows, we will shows that this element e_I is nontrivial in dimensions $d = 2, 3$. In particular, in dimension $d = 2$, we will give an explicit expression of this invariant, and it is identical with the fundamental class for the variety.

We begin first with a useful operation on extensions, which is called the disjoint sum. Given two compact metric spaces X_1 and X_2 with $X_1 \cap X_2 = \emptyset$, and two extensions $(\mathcal{E}_1, \pi_1)$, $(\mathcal{E}_2, \pi_2)$ of $\mathcal{K}$ by $C(X_1)$ on H_1 and $C(X_2)$ on H_2, respectively, an extension $(\mathcal{E}, \pi)$ of $\mathcal{K}$ by $C(X_1 \sqcup X_2)$ on $H_1 \oplus H_2$, which is called the disjoint sum of $(\mathcal{E}_1, \pi_1)$ and $(\mathcal{E}_2, \pi_2)$, is defined by

$$\mathcal{E} = \{T_1 \oplus T_2 + K : T_i \in \mathcal{E}_i,\ i = 1, 2,\ K \text{ is compact}\}$$
$$\pi(T_1 \oplus T_2 + K)|_{X_i} = \pi_i(T_i), \quad i = 1, 2.$$

where $\sqcup$ denotes disjoint union of spaces. From [10, Theorem 4.8], the above operation gives the following isomorphism

$$K_1(X_1 \sqcup X_2) \cong K_1(X_1) \oplus K_1(X_2). \tag{5.2}$$

Proposition 5.2 *Let I and J be homogeneous ideals, and $Z(I) \cap Z(J) = \{0\}$. If $[I]$, $[J]$ and $[I \cap J]$ are all essentially normal, then one has*

$$e_{I\cap J} = e_I \oplus e_J.$$

Proof For homogeneous ideals I, J, $I \cap J$, we consider three extensions

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I\rangle \xrightarrow{\pi_1} C(\mathcal{Z}(I)) \to 0,$$
$$0 \to \mathcal{K} \hookrightarrow C^*\langle J\rangle \xrightarrow{\pi_2} C(\mathcal{Z}(J)) \to 0,$$
$$0 \to \mathcal{K} \hookrightarrow C^*\langle I \cap J\rangle \xrightarrow{\pi_3} C(\mathcal{Z}(I \cap J)) \to 0.$$

Since $Z(I) \cap Z(J) = \{0\}$ and $Z(I \cap J) = Z(I) \cup Z(J)$, we have

$$\mathcal{Z}(I \cap J) = \mathcal{Z}(I) \bigsqcup \mathcal{Z}(J).$$

We will prove that the third extension is the disjoint sum of the first two. Considering the equality

$$[I \cap J]^{\perp} = [I]^{\perp} \oplus ([I] \ominus [I \cap J])$$

and for a polynomial p, setting

$$S_p^{(1)} = P_{[I]^{\perp}} M_p|_{[I]^{\perp}}, \ \ S_p^{(2)} = P_{[I\cap J]^{\perp}} M_p|_{[I\cap J]^{\perp}}, \ \ S_p^{(3)} = P_{[I]\ominus[I\cap J]} M_p|_{[I]\ominus[I\cap J]},$$

then we have

$$S_p^{(2)} = \begin{bmatrix} S_p^{(1)} & 0 \\ K_p & S_p^{(3)} \end{bmatrix}. \tag{5.3}$$

By assumption and Lemma 2.1, it is easy to check that K_p is compact, and $S_p^{(3)}$ is essentially normal. Hence

$$\sigma_e\left(S_{z_1}^{(2)}, \ldots, S_{z_d}^{(2)}\right) = \sigma_e\left(S_{z_1}^{(1)}, \ldots, S_{z_d}^{(1)}\right) \bigcup \sigma_e\left(S_{z_1}^{(3)}, \ldots, S_{z_d}^{(3)}\right).$$

From Theorem 5.1, it holds that

$$\sigma_e\left(S_{z_1}^{(2)}, \ldots, S_{z_d}^{(2)}\right) = \mathcal{Z}(I) \bigsqcup \mathcal{Z}(J)$$

and

$$\sigma_e\left(S_{z_1}^{(1)}, \ldots, S_{z_d}^{(1)}\right) = \mathcal{Z}(I).$$

Hence, we have

$$\mathcal{Z}(J) \subseteq \sigma_e\left(S_{z_1}^{(3)}, \ldots, S_{z_d}^{(3)}\right) \subseteq \partial\mathbb{B}_d.$$

For each polynomial $Q \in J$, it is easy to check that

$$Q\left(S_{z_1}^{(3)}, \ldots, S_{z_d}^{(3)}\right) = S_Q^{(3)} = 0.$$

This means that

$$\sigma_e\left(S_{z_1}^{(3)}, \ldots, S_{z_d}^{(3)}\right) \subseteq Z(J),$$

and hence we get that

$$\sigma_e\left(S_{z_1}^{(3)}, \ldots, S_{z_d}^{(3)}\right) = \mathcal{Z}(J).$$

From the decomposition (5.3), we see that

$$e_{I\cap J} = e_I \oplus \Lambda_J,$$

where $\Lambda_J \in K_1(\mathcal{Z}(J))$ is the K-homology element coming from the extension defined by $C^*(\mathrm{Id}, S^{(3)}_{z_1}, \dots, S^{(3)}_{z_d}) + \mathcal{K}$. The same reasoning shows that

$$e_{I\cap J} = \Lambda_I \oplus e_J,$$

where $\Lambda_I \in K_1(\mathcal{Z}(I))$. By (5.2),

$$K_1(\mathcal{Z}(I\cap J)) = K_1(\mathcal{Z}) \oplus K_1(\mathcal{Z}(J)).$$

This means that

$$\Lambda_J = e_J, \quad \Lambda_I = e_I,$$

and hence $e_{I\cap J} = e_I \oplus e_J$. The proof is complete. □

Remark Using the argument in the proof of Proposition 5.2, one can prove the following: If I and J are two homogeneous ideals satisfying $Z(I) \cap Z(J) = \{0\}$, and $[I]$, $[J]$ and $[I\,J]$ are all essentially normal, then

$$e_{IJ} = e_I \oplus e_J.$$

5.1 In dimension $d = 2$

Let I be a homogeneous ideal in two variables, and let $M = [I]$ be a graded submodule of 2-shift Hilbert module H^2_2. Then by Theorem 3.1, M is essentially normal, and hence gives an extension as in (5.1). This extension yields an odd K-homology element e_I for $\mathcal{Z}(I)$. We will prove $e_I \neq 0$, and give an explicit expression of this invariant.

Proposition 5.3 *Let $p(z_1, z_2) = \alpha z_1 + \beta z_2$, then we have $K_1(\mathcal{Z}(p)) = \mathbb{Z}$ and $e_{p^n} = -n$, where e_{p^n} denotes K-homology element determined by (5.1) when $I = p^n\mathbb{C}[z_1, z_2]$.*

Proof Firstly, consider the case $\alpha = 1, \beta = 0$, that is, $p(z_1, z_2) = z_1$. Since $\mathcal{Z}(z_1) = Z(z_1) \cap \partial\mathbb{B}_2 = \{(0, z_2) : |z_2| = 1\}$, and this set is a unit circle, we have $K_1(\mathcal{Z}(z_1)) \cong \mathbb{Z}$. With this identification, $e_{z_1^n} = \mathrm{Ind}\, S_2$ [9]. We claim that $\mathrm{Ind}\, S_2 = -n$. In fact, letting $h \in \ker S_2$, then $z_2 h \in [z_1^n]$. Write $h(z) = \sum a_{ij} z_1^i z_2^j$, then $i \geq n$ if $a_{ij} \neq 0$. This implies $h \in [z_1^n]$, and hence $h = 0$. Therefore, $\ker S_2 = 0$. Since

$$\ker\, S_2^* = [z_1^n]^\perp \cap \ker M^*_{z_2} = \mathrm{span}\{1, z_1, \dots, z_1^{n-1}\},$$

we have

$$\operatorname{Ind} S_2 = \dim\ker S_2 - \dim\ker S_2^* = -n.$$

Now let us consider the general case $p(z_1, z_2) = \alpha z_1 + \beta z_2$. Define a unitary transform $\mu : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\mu(z_1, z_2) = \left(\frac{\alpha z_1 + \beta z_2}{\sqrt{|\alpha|^2 + |\beta|^2}}, \quad \frac{-\overline{\beta} z_1 + \overline{\alpha} z_2}{\sqrt{|\alpha|^2 + |\beta|^2}} \right).$$

It is easy to check $\mu(Z(p)) = Z(z_1)$ and the transform μ induces a unitary operator $U : f \to f \circ \mu$ on H_2^2. Then the operator U maps $[z_1^n]^\perp$ onto $[p^n]^\perp$, and gives the following commutative diagram,

$$\begin{array}{ccccccccc} 0 \to & \mathcal{K} & \to & C^*\langle z_1^n \rangle & \to & C(\mathcal{Z}(z_1)) & \to & 0 \\ & \cong\downarrow U & & \cong\downarrow U & & \cong\downarrow \mu & & \\ 0 \to & \mathcal{K} & \to & C^*\langle p^n \rangle & \to & C(\mathcal{Z}(p)) & \to & 0. \end{array}$$

Hence $K_1(\mathcal{Z}(p)) \cong \mathbb{Z}$ and $e_{p^n} = -n$. □

We turn to any homogeneous polynomial $q(z_1, z_2)$. Then $q(z_1, z_2)$ can be decomposed as

$$q(z_1, z_2) = (\alpha_1 z_1 + \beta_1 z_2)^{\gamma_1} \dots (\alpha_n z_1 + \beta_n z_2)^{\gamma_n},$$

such that

$$Z(\alpha_i z_1 + \beta_i z_2) \cap Z(\alpha_j z_1 + \beta_j z_2) = \{0\}, \quad \text{if } i \neq j.$$

Combining Propositions 5.2 with 5.3 gives that

$$e_q = e_{(\alpha_1 z_1 + \beta_1 z_2)^{\gamma_1}} \oplus \cdots \oplus e_{(\alpha_n z_1 + \beta_n z_2)^{\gamma_n}} = (-\gamma_1, \dots, -\gamma_n).$$

For each homogeneous ideal I in $\mathbb{C}[z_1, z_2]$, the ideal I can be uniquely written as its Beurling form $I = pL$. By [37, Lemma 6.1], L is a finite codimensional ideal of $\mathbb{C}[z_1, z_2]$. This implies that $[p] = [I] \oplus N$ for some finite dimensional space N, and hence $[I]^\perp = [p]^\perp \oplus N$. Therefore, for quotient modules $[I]^\perp$ and $[p]^\perp$, the corresponding extensions (5.1) are weakly equivalent, and hence they are equivalent [9, Theorem 1.8]. This means $e_p = e_I$. We summarizes the above discussion to the following theorem.

Theorem 5.4 *In dimension $d = 2$, let $I = pL$ be the Beurling form of a homogeneous ideal I. Then the extension (5.1) determines one nontrivial element $e_p \in K_1(\mathcal{Z}(p))$, only depending on p. Furthermore, if $I = pL_1$, $J = qL_2$, and p and q have the same zero set, then $e_p = e_q$ if and only if $p = cq$, where c is a nonzero constant.*

5.2 In dimension $d = 3$

Let I be a homogeneous ideal in three variables, and let $M = [I]$ be a graded submodule of 3-shift Hilbert module H_3^2. Theorem 3.2 says that each graded submodule is essentially normal, and hence each graded quotient submodule is. This subsection will shows that each graded quotient submodule determines a nontrivial extension

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I\rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0, \tag{5.4}$$

and hence it gives rise to an odd nontrivial K-homology element e_I for the zero variety $\mathcal{Z}(I)$. This invariant contains important information on both the submodule and zero variety.

We begin with a well known fact. From the Universal Coefficient Theorem in [11], there exists a natural short exact sequence

$$0 \to Ext_{\mathbb{Z}}(K^0(X), \mathbb{Z}) \to Ext(X) \xrightarrow{\gamma_\infty} Hom(\widetilde{K}(X), \mathbb{Z}) \to 0.$$

Therefore, for a C^*-extension

$$0 \to \mathcal{K} \hookrightarrow \mathcal{A} \xrightarrow{\pi} C(X) \to 0, \tag{5.5}$$

if the extension is trivial (or split), then the image of the corresponding K-homology element by γ_∞ is also trivial. This implies the following result.

Lemma 5.5 *If the extension (5.5) is trivial, then for each natural number n and each Fredholm operator $A \in \mathcal{A} \otimes M_n$, we have* $\operatorname{Ind} A = 0$, *where M_n is the algebra of all complex $n \times n$ matrices.*

Firstly we show the K-homology element is nontrivial in the case of the principal submodule.

Proposition 5.6 *Let p be a homogeneous polynomial in three variables. Then the extension*

$$0 \to \mathcal{K} \hookrightarrow C^*\langle p\rangle \xrightarrow{\pi} C(\mathcal{Z}(p)) \to 0,$$

is nontrivial, where $C^*\langle p\rangle$ denotes the C^*-algebra $C^*\{\mathrm{Id}, S_1, S_2, S_3\}$ on $[p]^\perp$.

Proof It is easy to prove that there exists a pair of co-prime polynomials $q_1 = z_1 + \alpha_1 z_2 + \beta_1 z_3$, $q_2 = z_1 + \alpha_2 z_2 + \beta_2 z_3$, such that $Z(p) \cap Z(q_1) \cap Z(q_2) = \{0\}$. By Theorem 5.1, for the tuple $\{S_1, S_2, S_3\}$ acting on $[p]^\perp$, it holds that

$$\sigma_e(S_1, S_2, S_3) = \mathcal{Z}(p).$$

Since q_1, q_2 have no common zero point on $\sigma_e(S_1, S_2, S_3)$, the operator

$$A = \begin{bmatrix} S_{q_1} & S_{q_2} \\ -S_{q_2}^* & S_{q_1}^* \end{bmatrix}$$

is Fredholm. Clearly, $(1, 0)^T \in \ker A^*$ and hence $\ker A^* \neq \{0\}$, where $(1, 0)^T$ denotes the transpose of the vector $(1, 0)$. We claim $\operatorname{Ind} A \neq 0$. For the claim, it is enough to prove $\ker A = 0$. If $A(\xi_1, \xi_2)^T = 0$, then we have

(I) $q_1\xi_1 + q_2\xi_2 \in [p]$,
(II) $S^*_{q_1}\xi_2 = S^*_{q_2}\xi_1$.

Without loss of generality, we may assume that both ξ_1 and ξ_2 are homogeneous, and have the same degree. From the above (I), there exists a homogeneous polynomial ϕ such that $q_1\xi_1 + q_2\xi_2 = p\phi$. Since the ideal (q_1, q_2) generated by q_1, q_2 is prime, and $Z(p) \cap Z(q_1) \cap Z(q_2) = \{0\}$, this implies that there exist homogeneous polynomials h_1, h_2 such that $\phi = q_1h_1 + q_2h_2$, and hence $q_1\xi_1 + q_2\xi_2 = pq_1h_1 + pq_2h_2$. This gives that

$$q_1(\xi_1 - ph_1) = -q_2(\xi_2 - ph_2).$$

Since $GCD(q_1, q_2) = 1$, there exists a polynomial h such that

$$\xi_1 = ph_1 + q_2h, \quad \xi_2 = ph_2 - q_1h.$$

Writing $h = g_1 + g_2$ with $g_1 \in [p]$ and $g_2 \in [p]^\perp$, then

$$\xi_1 = (ph_1 + q_2g_1) + q_2g_2, \quad \xi_2 = (ph_2 - q_1g_1) - q_1g_2.$$

This means that

$$\xi_1 = S_{q_2}g_2, \quad \xi_2 = -S_{q_1}g_2.$$

Combining this equality with the above (II) yields

$$S^*_{q_1}S_{q_1}g_2 + S^*_{q_2}S_{q_2}g_2 = 0,$$

and hence $\xi_1 = S_{q_2}g_2 = 0$, $\xi_2 = -S_{q_1}g_2 = 0$. Therefore, $\ker A = 0$ and $\operatorname{Ind} A \neq 0$. The desired conclusion follows from Lemma 5.5. □

Combining Proposition 5.6 and Remark following Proposition 5.2, we have

Corollary 5.7 *For a homogeneous ideal I of $\mathbb{C}[z_1, z_2, z_3]$, let $I = pL$ be its Beurling form. If p is not a constant and $Z(p) \cap Z(L) = \{0\}$, then the extension*

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I\rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0$$

is nontrivial.

Let P be a prime ideal of $C[z_1, \ldots, z_d]$. The height of P, denoted by height(P), is defined as the maximal length l of any properly increasing chain of prime ideals

$$0 = P_0 \subset P_1 \cdots \subset P_l = P.$$

Since the polynomial ring $C[z_1, \ldots, z_d]$ is Noetherian, every prime ideal has finite height and the height of an arbitrary ideal is defined as the minimum of the heights of its associated prime ideals. For an ideal I, one has

$$\dim_{\mathbb{C}} Z(I) = n - l,$$

where $l = \text{height}(I)$ is the height of I (cf. [22,38]).

Lemma 5.8 *Let q be homogeneous polynomial in $\mathbb{C}[z_1, z_2, z_3]$, and let P be a homogeneous prime ideal of $\mathbb{C}[z_1, z_2, z_3]$ with $\text{height}(P) = 2$. If $q \notin P$, then $Z(P) \cap Z(q) = \{0\}$.*

Proof By the Lasker-Noether decomposition theorem [38, Vol. (I), p. 208], decompose

$$P + q\,\mathbb{C}[z_1, z_2, z_3] = \bigcap_{j=1}^{m} I_j \tag{5.6}$$

as its irredundant primary decomposition, where each I_j is P_j-primary for some prime ideal P_j. This implies $P \subseteq P_j$ for each j. If $P = P_j$ for some j, then the inclusion

$$P + q\,\mathbb{C}[z_1, z_2, z_3] \subseteq I_j \subseteq P_j = P$$

implies $q \in P$. This contradiction shows that P is strictly contained in P_j for each j. This means that each P_j has height 3, and hence each P_j is finite codimensional. Since each P_j is homogeneous, it follows that

$$Z(I_j) = Z(P_j) = \{0\}$$

for $j = 1, \ldots, m$. Therefore, by (5.6)

$$Z(P) \cap Z(q) = \bigcup_{j=1}^{m} Z(I_j) = \{0\},$$

completing the proof. □

Proposition 5.9 *For a homogeneous ideal I of $\mathbb{C}[z_1, z_2, z_3]$, if $\text{GCD}(I) = 1$, then the extension*

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I\rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0$$

is nontrival.

Proof From the Lasker-Noether decomposition theorem for polynomial ring [38, Vol.(I), p. 208], I can be decomposed as

$$I = I_0 \cap I_1 \cap \cdots \cap I_m,$$

where I_0 is a finite codimensional ideal, and each I_j is P_j-primary for some prime ideal P_j having height 2. Writing $J = I_1 \cap \cdots \cap I_m$, then

$$[I]^{\perp} = [J]^{\perp} + [I_0]^{\perp}.$$

Since $[I_0]^{\perp}$ is finite dimensional, without loss of generality, we may assume that I has decomposition $I = I_1 \cap \cdots \cap I_m$ such that each I_j is P_j-primary for some prime ideal P_j having height 2. Then one can choose a homogeneous polynomial p with degree 1 such that $Z(p) \cap Z(I) = \{0\}$. The choice is as follows: Let $V_j = P_j \cap \mathbb{H}_1$ for $j = 1, \ldots, m$, where $\mathbb{H}_1$ is the space of all homogeneous polynomials with degree 1. Since height$(P_j) = 2$, then V_j is a proper subspace of $\mathbb{H}_1$ for all j. By a simple linear algebra, $\cup_{j=1}^m V_j \subsetneqq \mathbb{H}_1$. Taking a nonzero $p \in \mathbb{H}_1 \setminus \cup_{j=1}^m V_j$, then by Lemma 5.8 such a p satisfies

$$Z(p) \cap Z(I) = \cup_{j=1}^m Z(p) \cap Z(I_j) = \cup_{j=1}^m Z(p) \cap Z(P_j) = \{0\}.$$

For this p, we see that S_p is Fredholm. We claim that $\mathrm{Ind} S_p \neq 0$. In fact, $1 \in \ker S_p^*$. Therefore it suffices to prove that $\ker S_p = 0$. Let $h \in [I]^{\perp}$ such that $S_p h = 0$, then $ph \in [I]$. Since I is homogeneous, without loss of generality, we assume that h is homogeneous, and hence $ph \in I$ by [22]. This means that $ph \in I_j$ for $j = 1, \ldots, m$. Using the characteristic space theory [12, Chap. 2], it is not difficult to show that $h \in I_j$ for each j, and hence $h \in I$. This says that $h = 0$, and therefore, $\ker S_p = 0$. The claim follows. The desired result comes from the claim and Lemma 5.5, completing the proof. □

Proposition 5.10 *Let I be a homogeneous ideal of $C[z_1, z_2, z_3]$, and let $I = pL$ be its Beurling form. If p is not a constant, then the extension*

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I \rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0$$

is nontrivial.

Proof From the Lasker-Noether decomposition theorem, the ideal L can be decomposed as

$$L = L_0 \cap L_1 \cap \cdots \cap L_m,$$

where L_0 is a finite codimensional ideal, and each L_j is P_j-primary for some prime ideal P_j having height 2. Writing $J = L_1 \cap \cdots \cap L_m$, then

$$\dim\ pJ/pL = \dim\ J/L_0 \cap J = \dim\,(J + L_0)/L_0 < \infty,$$

and hence $pJ = pL + R$ for some finite dimensional space R. Hence, we may assume that $L = L_1 \cap \cdots \cap L_m$.

Decompose p as the product of its prime factors, $p = p_1^{t_1} \cdots p_l^{t_l}$, and set

$$V_j = P_j \cap \mathbb{H}_1, \quad j = 1, \ldots, m, \quad W_i = p_i \mathbb{C} \cap \mathbb{H}_1, \quad i = 1, \ldots, l.$$

Then both V_j and W_i are proper subspaces of $\mathbb{H}_1$. By a simple linear algebra, the set $V = \cup_{j=1}^m V_j \bigcup \cup_{i=1}^l W_i$ is strictly contained in $\mathbb{H}_1$. Taking a nonzero polynomial $\phi \in \mathbb{H}_1 \setminus V$, Lemma 5.8 applies here to show that $Z(\phi) \cap Z(L) = \{0\}$. Since $\mathrm{GCD}(\phi, p) = 1$, the ideal (ϕ, p) generated by ϕ and p, has height 2 (see [12, Corollary 3.1.12]). Hence, as done in the proof of Proposition 5.9, one can choose a homogeneous polynomial ψ with degree 1 such that

$$Z(\psi) \cap Z(\phi) \cap Z(p) = \{0\}.$$

From the above discussion we have

$$Z(\psi) \cap Z(\phi) \cap Z(I) = \{0\}, \quad Z(\psi) \cap Z(\phi) \cap Z(p) = \{0\}. \tag{5.7}$$

Noticing that

$$[I]^\perp = [p]^\perp \oplus ([p] \ominus [I])$$

and for a polynomial q, putting

$$S_q^{(1)} = P_{[I]^\perp} M_q|_{[I]^\perp}, \ S_q^{(2)} = P_{[p]^\perp} M_q|_{[p]^\perp}, \ S_q^{(3)} = P_{[p]\ominus[I]} M_q|_{[p]\ominus[I]},$$

then we have

$$S_q^{(1)} = \begin{bmatrix} S_q^{(2)} & 0 \\ K_q & S_q^{(3)} \end{bmatrix}. \tag{5.8}$$

By Lemma 2.1 and Theorem 3.2, K_q is compact, and the operators $S_q^{(1)}, S_q^{(2)}, S_q^{(3)}$ are all essentially normal. For each polynomial $Q \in L$, since $I = pL$, we have $Q(S_{z_1}^{(3)}, S_{z_2}^{(3)}, S_{z_3}^{(3)}) = S_Q^{(3)} = 0$ and hence

$$\sigma_e\left(S_{z_1}^{(3)}, S_{z_2}^{(3)}, S_{z_3}^{(3)}\right) \subseteq \mathcal{Z}(L). \tag{5.9}$$

By (5.7), both the tuples $(S_\phi^{(1)}, S_\psi^{(1)})$ and $(S_\phi^{(2)}, S_\psi^{(2)})$ are Fredholm. By (5.9) and $Z(\phi) \cap Z(L) = \{0\}$, we see that $S_\phi^{(3)}$ is Fredholm, and the tuple $(S_\phi^{(3)}, S_\psi^{(3)})$ is Fredholm. Furthermore, applying [15, Proposition 11.1] gives that

$$\mathrm{Ind}\left(S_\phi^{(3)}, S_\psi^{(3)}\right) = 0.$$

From (5.8), it holds that

$$\mathrm{Ind}\left(S_\phi^{(1)}, S_\psi^{(1)}\right) = \mathrm{Ind}\left(S_\phi^{(2)}, S_\psi^{(2)}\right) + \mathrm{Ind}\left(S_\phi^{(3)}, S_\psi^{(3)}\right) = \mathrm{Ind}\left(S_\phi^{(2)}, S_\psi^{(2)}\right).$$

By [14], and using the same reasoning as in the proof of Proposition 5.6, we have

$$\mathrm{Ind}\left(S_\phi^{(2)}, S_\psi^{(2)}\right) = \mathrm{Ind}\begin{bmatrix} S_\phi^{(2)} & S_\psi^{(2)} \\ -S_\psi^{(2)*} & S_\phi^{(2)*} \end{bmatrix} \neq 0,$$

and hence

$$\mathrm{Ind}\left(S_\phi^{(1)}, S_\psi^{(1)}\right) = \begin{bmatrix} S_\phi^{(1)} & S_\psi^{(1)} \\ -S_\psi^{(1)*} & S_\phi^{(1)*} \end{bmatrix} \neq 0.$$

The desired result follows from Lemma 5.5, completing the proof. □

Combining Propositions 5.9 and 5.10, we reach at the main result in this subsection.

Theorem 5.11 *Let I be a homogeneous ideal of $\mathbb{C}[z_1, z_2, z_3]$ with infinite codimension, then the extension*

$$0 \to \mathcal{K} \hookrightarrow C^*\langle I\rangle \xrightarrow{\pi} C(\mathcal{Z}(I)) \to 0$$

is nontrivial.

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