Supplementary information for Letter “Oscillation suppression and synchronization: Frequencies determine the role of control with time delays”

Wei Lin, Yang Pu, Yao Guo, and Jürgen Kurths

1School of Mathematical Sciences, Centre for Computational Systems Biology, Fudan University, Shanghai 200433, China, and Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, China.

2Potsdam Institute for Climate Impact Research, D-14412 Potsdam, and Department of Physics, Humboldt University of Berlin, D-12489 Berlin, Germany.

In this supplementary information, we perform a rigorous and systematic analysis of a transcendental characteristic equation with complex coefficients, thereby establishing necessary and sufficient conditions for guaranteeing the stability of the controlled supercritical-Hopf-bifurcation model with a time delay feedback. Besides, we expatiate, respectively, on the several transformations used for the oscillation suppression in the FitzHugh-Nagumo model and for the synchronization of a periodic orbit in a complex system of $N$ connected units with time delay couplings.

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References
I. PRELIMINARY FORMULATIONS

A linearization at the equilibrium $z_0 = 0$ of the controlled supercritical-Hopf-bifurcation model with a time delay feedback [refer to model (2) in Letter] yields:

$$\dot{z}(t) = (a + id)z(t) + ke^{id}z(t - \tau),$$  \tag{S1}

where $\tau > 0$ is a time delay of the feedback term and $ke^{id}$ is a complex-valued control gain. Actually, Eq. (S1) is the same as the linear model (3) which is in a matrix form in Letter. Thus, according to the theory in [1], the asymptotical stability of Eq. (S1) is determined by its characteristic equation

$$\lambda - (a + id) - ke^{id}e^{-\lambda \tau} = 0,$$  \tag{S2}

which is the same as Eq. (4) in Letter. We import the following transformations of the argument and complex parameters: $Z = \lambda \tau - a \tau - id\tau$ and $\beta = be^{i\psi}$, which have been defined in Letter. Here, $b = -k\tau e^{-i\psi}$ and $\alpha = \psi - d\tau (\mod \pi)$, so that $b \in \mathbb{R}$ and $\alpha \in [0, \pi)$. Hence, Eq. (S2) is transformed equivalently into an equation with respect to the newly defined argument $Z$:

$$h(Z) \triangleq Z + \beta e^{-Z} = 0,$$  \tag{S3}

which is provided as Eq. (5) in Letter. Let $A = a\tau$. Then, from $Z = \lambda \tau - a \tau - id\tau$, it follows that all the eigenvalues of Eq. (S2) are on the left half-plane if and only if every root of Eq. (S3) in the complex plane is on the left side of the line $Z = -A$, that is, for any root $Z = Z_0$ of Eq. (S3), $Re(Z_0) < -A$.

Substitution of $Z = x + iy$ into Eq. (S3) implies the equivalence between $Re(Z) < -A$ and $x < -A$. Equating the real and imaginary parts separately yields:

$$\begin{cases} x = -be^{-x} \cos(y - \alpha), \\ y = be^{-x} \sin(y - \alpha). \end{cases}$$  \tag{S4}

We are first to give an elementary description of the root distribution of Eq. (S4). For this purpose, we consider two separate cases, viz. $\alpha \neq 0$ (i.e. $\beta$ is non-real) and $\alpha = 0$ (i.e. $\beta$ is real) in this section. It is noted that the particular case of $\alpha = 0$ and $A = 0$ has been discussed in the Appendix of [1] and references therein. However, a more general case of $\alpha = 0$ and $A \geq 0$ is considered here.

A. Non-real-valued $\beta$

Throughout this subsection, $\beta$ is assumed to be non-real, so that $\alpha \in (0, \pi)$. It thus is easy to validate $y \neq 0$ since $y = 0$ leads to $b = 0$ and $\beta = 0$, which is not included in the present case. Hence, direct operation of the two equations in Eq. (S4) gives:

$$\begin{cases} F(x, y) \triangleq x + y \cot(y - \alpha) = 0, \\ y \csc(y - \alpha) = be^{-\cot(y - \alpha)}. \end{cases}$$  \tag{S5}

For consideration of a well-defined $\cot(y - \alpha)$, it is required (C1): $y \neq \alpha + p\pi$ for $p \in \mathbb{Z}$. However, the straight lines $L_p : y = \alpha + p\pi$ in the $x$-$y$ plane have particular meanings for the curve represented by $F(x, y) = 0$, which will be illustrated later.

First, derivative of both sides of $F(x, y)$ with respect to $y$ yields:

$$\frac{\partial F}{\partial y} = \frac{-y + \sin(y - \alpha) \cos(y - \alpha)}{\sin^2(y - \alpha)},$$

This quantity is nonzero if (C2): $y \neq \sin(y - \alpha) \cos(y - \alpha)$, i.e. $2y \neq \sin(2(y - \alpha))$, is satisfied. Under the conditions (C1)-(C2), the Implicit Function Theorem [2] could be implied to $F(x, y) = 0$. It thus is assured the existence of a locally differentiable function $y = \psi(x)$ satisfying $\psi(x_0) = y_0$ and $F(x, \psi(x)) = 0$ in the neighborhood of $x = x_0$. Hence, $\psi(x)$ satisfies an initial value problem of the ordinary differential equation:

$$\begin{cases} \frac{dy}{dx} = \frac{\sin^2(y - \alpha)}{y - \sin(y - \alpha) \cos(y - \alpha)}, \\ \psi(x_0) = y_0. \end{cases}$$  \tag{S6}
The function on the right side of the first equation in problem (S6) fulfills a Lipschitz condition when \(2y \neq \sin 2(y - \alpha)\). It thus from the general theory of ordinary differential equations (ODEs) [3, 4] follows that the initial value problem (S6) can be uniquely solved in the neighborhood of \(x = x_0\) as the initial value satisfies \(2y_0 \neq \sin 2(y_0 - \alpha)\). In order to discuss the maximal interval of existence (MIE) for each solved solution, we need the following result.

**Lemma I.1** The root of the equation \(2y = \sin 2(y - \alpha)\) is uniquely existent for \(\alpha \in (0, \pi)\).

**proof.** Consider the function \(\eta(y) \equiv 2y - \sin 2(y - \alpha)\). On the one hand, note that \(\eta(y) \to \pm \infty\) as \(y \to \pm \infty\). Hence, by the continuity of \(\eta(y)\), the zero of \(\eta(y)\) is existent. On the other hand, the derivative \(\eta'(y) = 2 - 2 \cos 2(y - \alpha) \geq 0\), where the equality is valid only when \(y_s = s\pi + \alpha\) for \(s \in \mathbb{Z}\). However, none of \(y_s\) is the zero of \(\eta(y)\) as \(\alpha \in (0, \pi)\), so that the zero of \(\eta(y)\) is unique. Therefore, we complete the proof. \(\blacksquare\)

With Lemma I.1, we get that, in problem (S6), \(dy/dx > 0\) if \(y > y^*\) and \(dy/dx < 0\) if \(y < y^*\), where \(y = y^* = y^*(\alpha)\) is the unique root of \(2y = \sin 2(y - \alpha)\) for a given \(\alpha \in (0, \pi)\). This means that the solutions of problem (S6) are strictly increasing when their initial values satisfy \(y_0 > y^*\) and strictly decreasing as \(y_0 < y^*\). Also note that the first equation in problem (S6) has constant solutions corresponding to the straight line \(s^\alpha\) passing through \((x_0, y_0)\). Again, in light of the general theory of ODEs [3, 4], the MIE for every solution of problem (S6) can be extended to \(\pm \infty\) as the initial value locates between the line \(L_p\) and \(L_{p-1}\) for \(p \neq 0\); however, the MIE can be extended only to \(+\infty\) as the initial value locates between \(L_0\) and \(L_{-1}\) and on the right side of the line \(x = x^*\) where \(x^* = y^* \cot(y^* - \alpha)\).

Now, from the uniqueness of every solution starting from \((x_0, y_0)\), it follows that each trajectory of the solution on the entire MIE corresponds to a single curve branch, denoted by \(\Gamma_p\), in between the straight lines \(L_p\) and \(L_{p-1}\), and that these straight lines indeed become asymptotes of the branches. We thus obtain the curve of \(F(x,y) = 0\), denoted by \(\Gamma = \bigcup_{p=\pm \mathbb{Z}} \Gamma_p\), and arrive to the following specific description.

**Lemma I.2** For \(p \in \mathbb{Z}^+\) (\(p \in \mathbb{Z}^-\), resp.), the branch \(\Gamma_p\) of the curve \(\Gamma\) is strictly increasing (decreasing, resp.) and transversely crosses the \(y\)-axis at \((0, y_p)\) with \(y_p = p\pi - \pi/2\). In addition, for \(p = 0\), \(\Gamma_0\) is not monotonic but concave to the right, thereby crossing the \(y\)-axis at both \((0, 0)\) and \((0, y_0)\) with \(y_0 = \alpha - \pi/2\). The coordinate of the vertex of \(\Gamma_0\) is \((-\cos^2(y^* - \alpha), y^*)\), which merges with \((0, y_0)\) into \((0, 0)\) as \(\alpha = \pi/2\).
for all \( \alpha \in (0, \pi) \). Then, with changing \( \alpha \), the vertex of \( I_0 \) varies on a circle with a center at \((-1/2, 0)\) and a radius of 1/2. In particular, with the increase of \( \alpha \) from 0 to \( \pi \), the vertex moves clockwise on the circle from \((-1, 0)\) to itself. According to Lemma I.2, the vertex passes by \((0, 0)\) on the circle as \( \alpha = \pi/2 \).

The analyses performed above manifest that all the roots of Eq. (S4) are located on the curve \( \Gamma \). In what follows, we need to investigate how many roots are there and how the roots move along the branches of \( \Gamma \) with changing \( b \).

This motivates us to consider the second equation in Eq. (S5). Actually, every root of the second equation could be regarded as an intersection point of the two curves of functions \( u_1(y) = y \csc(y - \alpha) \) and \( u_2(y) = be^{y \cot(y - \alpha)} \), where \( b \neq 0 \) could be viewed as a parameter determining the locus of the intersection point. Once the intersection is found for a given \( b \), the corresponding root of Eq. (S4) can be directly obtained through substituting the \( y \) coordinate of the intersection point into the first equation of Eq. (S5).

Using a method similar to the discussion for the curve \( \Gamma \), we know that both curves represented by \( u_1(y) \) and \( u_2(y) \) have an infinite number of branches separated by the vertical lines \( V_p : y_p = \alpha + p\pi \) in the \( y-u \) plane for \( p \in \mathbb{Z} \). More concretely, we get the following result.

**Lemma I.3** If \( b > 0 \), roots of the equation \( u_1(y) = u_2(y) \) are only existent in the intervals \( I_p = (\alpha + p\pi, \alpha + (p + 1)\pi) \) for \( p = 2q - 2 \) and \( p = -2q - 1 \) \((q \in \mathbb{N})\). Conversely, if \( b < 0 \), roots are only existent in the intervals \( I_p \) for \( p = 2q - 1 \) and \( p = -2q \) \((q \in \mathbb{N})\). Each interval contains only one root for both cases.

**proof.** We first write \( \zeta(y) = u_1(y) - u_2(y) \), and, without loss of generality, suppose \( b > 0 \), \( p = 2q - 2 \) and \( p = -2q - 1 \) \((q \in \mathbb{N})\). Similar arguments can be performed for the case of \( b < 0 \).

On the one hand, \( \zeta(y) \) goes to \( +\infty \) as \( y \to (\alpha + (p + 1)\pi)^- \), and goes to \( -\infty \) as \( y \to (\alpha + p\pi)^+ \). Here, the divergence of \( u_2 \) is exponentially faster than that of \( u_1 \) as \( y \to (\alpha + p\pi)^+ \). Thus, from the continuity of \( \zeta(y) \) in \( I_p \), it follows that the zero of \( \zeta(y) \) is existent in \( I_p \).

On the other hand, derivative of \( \zeta(y) \) yields:

\[
\zeta'(y) = \frac{\sin(y - \alpha) - y \cos(y - \alpha)}{\sin^2(y - \alpha)} - be^{y \cot(y - \alpha)} \frac{y + \sin(y - \alpha) \cos(y - \alpha)}{\sin^2(y - \alpha)}.
\]

This derivative at \( y = y^0 \) which is the zero of \( \zeta(y) \) becomes:

\[
\zeta'(y^0) = \frac{1}{\sin(y^0 - \alpha)} \left\{ \left[ \frac{y^0}{\sin(y^0 - \alpha)} - \cos(y^0 - \alpha) \right]^2 + \sin^2(y^0 - \alpha) \right\}.
\]

Since \( y^0 - \alpha \in (p\pi, (p + 1)\pi) \), the sign of \( \sin(y^0 - \alpha) \) in each \( I_p \) keeps unchangeable. Moreover, \( \sin^2(y^0 - \alpha) \neq 0 \) in each \( I_p \), so that the sign of \( \zeta'(y^0) \), which is the same as that of \( \sin(y^0 - \alpha) \), is invariant in \( I_p \). This implies the uniqueness of the zero in each \( I_p \); otherwise, the sign of \( \zeta'(y^0) \) for different \( y^0 \) in the same \( I_p \) is changeable. Therefore, we approach the conclusion of the lemma on the roots of the equation \( u_1(y) = u_2(y) \).

**Lemma I.4** If \( b \neq 0 \), the roots of the equation \( u_1(y) = u_2(y) \) are uniquely existent in the interval \( I_{-1} = (\alpha - \pi, \alpha) \).

**proof.** It is clear that, for any \( b \neq 0 \), \( \zeta(y) = u_1(y) - u_2(y) \) goes to \( -\infty \) as \( y \to \alpha^- \), and goes to \( +\infty \) as \( y \to \alpha^+ \). This implies the existence of a zero \( y^0 \in I_{-1} \). Moreover, the uniqueness of the zero follows from the unchangeable sign of \( \zeta'(y^0) \), which has been validated in the proof of Lemma I.3.

From Lemmas I.3 and I.4, we know that there are an infinite number of roots for the equation \( u_1(y) = u_2(y) \), which correspond to the intersection points of the curve branches in between the vertical lines \( V_p \) and \( V_{p+1} \) for \( p \in \mathbb{Z} \). More importantly, in between every pair of the vertical lines, the intersection point is unique. Fig. S-2 shows some of the branches and their intersection points for the cases of \( b = 1 \) and \( b = -1 \).

Once the intersection points are all determined for a given \( b \neq 0 \), the roots of Eq. (S5) can be obtained. In particular, each intersection point corresponds to only one root which locates on one of the branches of the curve \( \Gamma \).

Next, we need to show how the root moves on each branch of \( \Gamma \) for changing \( b \). For this purpose, we suppose \( Z \) to be a root of Eq. (S3), and so that \( Z \) could be regarded as a function of \( b \). We thus are to check
the signs of the real and the imaginary parts of $Z'(b)$, respectively. Note that $Z = -\beta e^{-Z} = -be^{\alpha-Z}$. Then, $Z'(b) = Z|b - Z : Z'(b)$, which further implies

$$Z'(b) = \frac{Z}{b(1 + Z)} = \frac{1}{b} \left( 1 - \frac{1}{1 + Z} \right) = \frac{1}{b} \left( 1 - \frac{1 + x - iy}{|1 + Z|^2} \right),$$

where $Z(b) = x(b) + iy(b)$ which we have set above. Equating the real and the imaginary parts, respectively, yields:

$$x'(b) = \frac{1}{b} \cdot \frac{(x + \frac{1}{2})^2 + y^2 - \frac{1}{4}}{(1 + x)^2 + y^2}, \quad y'(b) = \frac{1}{b} \cdot \frac{y}{(1 + x)^2 + y^2}.$$

It is clear that $y'(b) > 0$ along the branches $\Gamma_p$ for $b > 0$, $y > 0$ and $p = 2q - 1$ ($q \in \mathbb{N}$), and that $y'(b) < 0$ along the branches $\Gamma_p$ for $b < 0$, $y > 0$ and $p = 2q$ ($q \in \mathbb{N}$). Also it is easy to get the sign of $y'(b)$ as $y > 0$ for different $b \neq 0$. Because all the branches are uniquely obtained, the movement direction of the roots on the branches $\Gamma_p$ ($p \neq 0$) can be determined consequently, as shown in Fig. S.1.

As for the concave branch $\Gamma_0$, it crosses a circle with a center at $(-1/2, 0)$ and a radius of $1/2$ at, respectively, the origin and and the vertex of $\Gamma_0$. In particular, at the origin $b = 0$, and at the vertex $b = b_0 = \cos(y' - \alpha) \cdot e^{-\cos(y' - \alpha)}$, \hspace{1cm} (S8)

where $\cos(y' - \alpha)$ is positive for $0 < \alpha < \pi/2$ and negative for $\pi/2 < \alpha < \pi$. In addition, the numerator on the right side of $x'(b)$ is negative when $\Gamma_0$ is inside the circle but positive when it is outside. Thus, it is not hard to validate that, whether or not $b_0$ is positive or negative, $x'(b) < 0$ for $b < b_0$ and $x'(b) > 0$ for $b > b_0$.

Altogether, we summarize the above results into the following proposition.

**Proposition 1.5** Case I “$b > 0$”: For $p = 2q - 1$ ($q \in \mathbb{N}$), the single root on each branch $\Gamma_p$ that starts from $(-\infty, (p - 1)\pi + \alpha)$ when $b = 0$ and goes to $(+\infty, p\pi + \alpha)$ when $b = +\infty$, crossing the $y$-axis at the point $(0, \alpha + p\pi - \pi/2)$ when $b = a + p\pi - \pi/2$. For $p = 2q$ ($q \in \mathbb{N}$), the root on each branch starts from $(-\infty, p\pi + \alpha)$ when $b = 0$ and goes to $(+\infty, (p - 1)\pi + \alpha)$ when $b = +\infty$, crossing the $y$-axis at the point $(0, \alpha + p\pi - \pi/2)$ when $b = \alpha - p\pi + \pi/2$.

Case II “$b < 0$”: For $p = 2q$ ($q \in \mathbb{N}$), the single root on each branch $\Gamma_p$ of Eq. (S4), denoted by $(x_p^0, y_p^0)$, that starts from $(+\infty, p\pi + \alpha)$ when $b = -\infty$ and goes to $(-\infty, (p - 1)\pi + \alpha)$ when $b = 0$. It crosses the $y$-axis at the point $(0, \alpha + p\pi - \pi/2)$ when $b = -\alpha - p\pi + \pi/2$. Moreover, for $p = -2q + 1$ ($q \in \mathbb{N}$), the root starts from $(+\infty, (p - 1)\pi + \alpha)$ when $b = -\infty$ and goes to $(-\infty, p\pi + \alpha)$ when $b = 0$, crossing the $y$-axis at the point $(0, \alpha - p\pi - \pi/2)$ when $b = -\alpha + p + \pi/2$.
Case III “b ≠ 0 and “p=0”: The single root on the branch $\Gamma_0$ starts from $(+\infty, \alpha)$ when $b = -\infty$ and goes to $(+\infty, -\pi + \alpha)$ when $b = +\infty$, crossing the y-axis twice, viz., $(0,0)$ when $b = 0$ and $(0, \alpha - \pi/2)$ when $b = -\alpha + \pi/2$. The zero approaches the leftmost position ($-\cos^2(y^* - \alpha), y^*$) when $b = b_0$ as defined in (S8).

B. Real-valued $\beta$

In this subsection, we discuss the case of real-valued $\beta$. Then, letting $\alpha = 0$ leads Eq. (S4) to

$$\begin{align*}
  x &= -be^{-x} \cos y, \\
  y &= be^{-x} \sin y.
\end{align*}$$

Clearly, the second equation in Eq. (S9) has a zero root, i.e. $y = 0$. Therefore, any root of $x = -be^{-x}$ is the real root of Eq. (S4) and vice versa. The real root of $x = -be^{-x}$ can be obtained from the intersection points between the curves represented by $v_1(x) = x$ and $v_2(x) = -be^{-x}$, as shown in Fig. S-3. Theoretically, we have the following results.

![FIG. S-3. The intersection points between the curves represented by $v_1(x) = x$ (blue) and $v_2(x) = -be^{-x}$ where the parameter $b = -1$ (red), $b = 1/e$ (black) and $b = 1$ (green).](image)

**Proposition I.6** For $b < 0$, Eq. (S4) has only one positive and real root, denoted by $x_b^+$. With increasing $b$ from $-\infty$ to $0^-$, $x_b^+$ decreases from $+\infty$ to $0^-$. For $b = 0$, the root is zero. For $0 < b < 1/e$, Eq. (S4) has two negative and real roots, denoted by $x_{1b}$ and $x_{2b}$. These two roots merge into $-1$ as $b = 1/e$. For $b > 1/e$, Eq. (S4) has no real roots.

**proof.** Write $\rho(x) = v_1(x) - v_2(x)$. Then, it is clear that, for $b \leq 0$, $\rho(x)$ goes to $+\infty$ as $x \to +\infty$ and $\rho(0) = b \leq 0$. Thus, from the continuity of $\rho(x)$, it has at least one zero on $[0, +\infty)$. It is easy to validate that $\rho(x)$ has no zero on $(-\infty, 0)$. Moreover, $\rho'(x) = 1 - be^{-x} > 0$ for $b \leq 0$. Therefore, the zero of $\rho(x)$ is uniquely existent on $(-\infty, +\infty)$ for $b \leq 0$, and, in particular, $x = 0$ is the zero of $\rho(x)$ as $b = 0$.

For the case of $b > 0$, any root of $\rho(x)$ is negative since $v_2(x) < 0$. $\rho'(x) = 1 - be^{-x} = 0$ gives $\rho(x) < 0$ as $x \in \mathcal{F}_1$, and $\rho(x) > 0$ as $x \in \mathcal{F}_2$. When $0 < b < 1/e$, we have $\rho(\ln b) < 0$ and $\rho(\pm \infty) = +\infty$. Therefore, from the continuity of the function $\rho(x)$, it follows that there is a zero of $\rho(x)$ in each of $\mathcal{F}_{1,2}$ when $0 < b < 1/e$.

More accurately, write $\rho(x, b) = v_1(x) - v_2(x)$. Thus, the derivative $\rho_x(x, b) = 1 - be^{-x}$, which together with $\rho_x(x, b) = 0$ implies $x = -1$ and $b = 1/e$. Hence, $\rho(-1, 1/e) = 0$. Note that $\rho_y(-1, 1/e) = e \neq 0$. Then, according to the Implicit Function Theorem [2], there exists a function $b = b(x)$ in the neighborhood of $x = -1$ such that $\rho(x, b(x)) = 0$ and $b(-1) = 1/e$. Furthermore,

$$\frac{d\rho(x, b(x))}{dx} = 1 + b'(x)e^{-x} - b(x)e^{-x} = 0,$$
which further yields $b'(-1) = 0$ and $b''(-1) = -b(-1) < 0$. This actually implies that the graph of $b = b(x)$ describes the variation of the two zeros of $\rho(x)$ with the change of $0 < b < 1/e$. Particularly, the two zeros merge into $x = -1$ as $b = 1/e$ and the real zero disappears as $b > 1/e$. \[ \]

Assume $y \neq 0$. We thus are able to investigate the non-real roots of Eq. (S9) as well as Eq. (S4). Eq. (S9) thus can be transformed equivalently into

$$\begin{cases} \displaystyle x + y \cot y = 0, \\ y \csc y = b e^{\cot y}. \end{cases}$$

First, it is easy to see that all the roots appear symmetrically with respect to the $y$-axis. Then, using the arguments as those performed in the preceding subsection, we know that all the roots are located on an infinite number of the branches represented by the first equation in Eq. (S10). In particular, in addition to those branches, denoted by $\Gamma_p$ ($p \neq 0$) monotonically transversing the $y$-axis, the branch, denoted by $\Gamma_0$, is concave to the right and occupies a strip with a wide of $2\pi$. Figure S-4 shows some of these branches and the movement direction of the root on each branch with the change of $b$. The concave branch here is slightly different from the concave branch presented for the case of non-real-valued $\beta$ and shown in Fig. S-1. Theoretically, we approach the following results on the character of the non-real roots of Eq. (S4) for any real-valued $\beta$.

![Figure S-4](image-url)

**Fig. S-4.** Some of the branches $\Gamma_p$ ($p \in \mathbb{Z}$) of the curve represented by the first equation in Eq. (S10). All the real roots of Eq. (S4), obtained in Proposition I.6, are located on the $x$-axis. The arrows represent the movement direction of the roots on the branches and the $x$-axis with the increase of $b$.

**Proposition I.7** Case I “$b < 0$”: For any integer $p = 2q + 1$ ($q \in \mathbb{Z}$), there is one root of Eq. (S4) on each $\Gamma_p$. As $b$ changes from $-\infty$ to 0, this root moves from $(-\infty, (2p + 2)\pi)$ to $(-\infty, (2p + 1)\pi)$, crossing the $y$-axis from right to left at $(0, 2p\pi + 3\pi/2)$ when $b = -2p\pi - 3\pi/2$.

Case II “$b > 0$”: For any integer $p = 2q$ ($q \in \mathbb{Z}$, $q \neq 0$), there is one root of Eq. (S4) on each $\Gamma_p$. As $b$ changes from 0 to $+\infty$, this root moves from $(-\infty, 2p\pi)$ to $(+\infty, (2p + 1)\pi)$, crossing the $y$-axis from left to right at $(0, 2p\pi + \pi/2)$ when $b = 2p\pi + \pi/2$.

Case III “$b > 1/e$”: There are two roots symmetrically locating on the concave branch $\Gamma_0$. These two zeros merge into $(-1, 0)$ as $b$ goes to $1/e$ and tends towards, respectively, $(+\infty, \pm \pi)$ as $b$ goes to $+\infty$. They cross the $y$-axis at $(0, \pm \pi/2)$ when $b = \pi/2$.

Case IV “$b < 1/e$”: There is one real root of Eq. (S4) on the $x$-axis, moving from $(+\infty, 0)$ to $(-1, 0)$ with the increase of $b$ from $-\infty$ to $1/e$. There is the other real root on the $x$-axis, moving from $(-\infty, 0)$ to $(-1, 0)$ with the increase of $b$ from $0^+$ to $1/e$. When $b > 0$, two roots simultaneously appear, which have been described in Proposition I.6, and finally merge into $(-1, 0)$ as $b = 1/e$.

**Remark I.8** As a matter of fact, part of the arguments on the two cases discussed in the above section could be found briefly in [5]; however, for the sake of accuracy and integrity, we provide rigorous analyses and then establish systematic results here.
II. NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY

Based on the above-obtained results on the root distribution of Eq. (S4) as well as Eq. (S3), we, in this section, are to establish necessary and sufficient (NS) conditions under which all the roots of Eq. (S4) satisfy \( \text{Re} [Z] < -A = -\alpha \tau \), i.e., \( x < -A \), and then the asymptotical stability of the controlled system (S1) is assured. For this purpose, we consider the two cases of \( \beta \) discussed in the preceding section, and finally summarize the established conditions into the NS conditions.

First, let \( x = -A \) in Eq. (S4). Then, we get

\[
\begin{aligned}
-A &= -b e^A \cos(y - \alpha), \\
y &= b e^A \sin(y - \alpha).
\end{aligned}
\]

(S11)

Clearly, \( b = -kte^{-\alpha t} = 0 \) implies \( k = 0 \) and \( x = y = 0 \) in Eq. (S11). From \( x = 0 < -A = -\alpha \tau \), it follows that \( a < 0 \) becomes the stability condition when \( k = 0 \). In the following discussion, we assume that \( b \neq 0 \). Hence, Eq. (S11) can be transformed equivalently into

\[
\begin{aligned}
A^2 + y^2 &= b^2 e^{2A} \\
\cos(y - \alpha) &= \frac{A}{b e^A}, \\
0 &\leq b y \sin(y - \alpha).
\end{aligned}
\]

Further calculation of the above equations yields:

\[
\begin{aligned}
y &= \pm \sqrt{b^2 e^{2A} - A^2}, \\
y &= \alpha \pm \arccos \left( \frac{A}{b e^A} \right) + 2p\pi, \quad p \in \mathbb{Z}, \\
0 &\leq b y \sin(y - \alpha).
\end{aligned}
\]

(S12)

It is clear that the first equation in Eq. (S12) is solvable only if (C-I): \( b^2 e^{2A} - A^2 \geq 0 \), i.e., \( |k| \geq |a| \) in the controlled system (S1). This situation will be discussed in the following subsections; however, we first deal with Eq. (S4) under the condition (C-II): \( b^2 e^{2A} - A^2 < 0 \), and then establish the following two propositions.

**Proposition II.1** If \( A < 0 \) and \( b^2 e^{2A} < A^2 \) (i.e., \( a < 0 \), \( \tau > 0 \), and \( |k| < -a \)), any root \((x, y)\) of Eq. (S4) satisfies \( \text{Re} [Z] = x < -A \), so that the controlled system (S1) is asymptotically stable when \( a < 0 \), \( \tau > 0 \), and \( |k| < -a \).

**proof.** Suppose that there exists a solution of Eq. (S4), denoted by \((x^0, y^0)\), such that \( x^0 \geq -A > 0 \). Then we have

\[
(x^0)^2 e^{2A} \leq [(x^0)^2 + (y^0)^2] e^{2A} = b^2 e^{-2y^0} e^{2A} < A^2 e^{-2y^0},
\]

which implies

\[
(x^0)^2 < A^2 e^{-2(y^0 + A)} \leq A^2,
\]

where the second inequality is due to \( x^0 + A \geq 0 \). However, \((x^0)^2 < A^2 \) contradicts the supposition, which finally implies the completion of the proof.

**Proposition II.2** If \( A > 0 \) and \( b^2 e^{2A} < A^2 \) (i.e., \( a > 0 \), \( \tau > 0 \), and \( |k| < a \)), then there exists at least one root \((x^0, y^0)\) of Eq. (S4), such that \( \text{Re} [Z^0] = x^0 > -A \). Thus, the controlled system (S1) is unstable when \( a > 0 \), \( \tau > 0 \), and \( |k| < a \).

**proof.** For a given \( A > 0 \), we consider a map \( \Psi : S \subset \mathbb{C} \to \mathbb{C} \). Here, the set \( S \triangleq [-A, A] \times [-A, A] \) is bounded, closed and convex, and the map

\[
\Psi(x, y) \triangleq (-be^{-x} \cos(y - \alpha), be^{-x} \sin(y - \alpha))
\]

follows from Eq. (S11). Notice the conditions \( A > 0 \) and \( b^2 e^{2A} < A^2 \) assumed in Proposition II.2. We thus get \( |be^{-x}| \leq |b| e^A < A \) for any \((x, y) \in S\), so that

\[
| -be^{-x} \cos(y - \alpha) | < A, \quad |be^{-x} \sin(y - \alpha) | < A,
\]
for any \((x, y) \in S\). This manifests that \(\Psi\) maps \(S\) to itself, i.e., \(\Psi: S \rightarrow S\). Also it is obvious that the map \(\Psi\) is continuous. Therefore, by virtue of the well-known Brouwer fixed-point theorem [6], we conclude that there exists a point \((x^0, y^0) \in S\) such that \((x^0, y^0) = \Psi(x^0, y^0)\). This means that Eq. (S4) at least possesses a root \((x^0, y^0)\) with \(x^0 \in [-A, A]\). If \(x^0 = -A\), we get

\[
A^2 \leq A^2 + (y^0)^2 = (x^0)^2 + (y^0)^2 = b^2e^{-2x^0} = b^2e^{2A} < A^2,
\]

for some \(y^0 \in [-A, A]\). However, this is a contradiction, which consequently implies \(x^0 > -A\) as well as the completion of the proof.

Now, under the condition (C-1): \(b^2e^{2A} - A^2 \geq 0\), we divide the following discussions into three cases, viz., the case of transverse branches for non-real-valued \(\beta\), the case of a concave branch for non-real-valued \(\beta\), and the case of all branches for real-valued \(\beta\).

A. Along the transverse branches for non-real-valued \(\beta\)

Here, we consider the case of all transverse branches \(\Gamma_{p+1}\) for non-real-valued \(\beta\), i.e., \(p \in \mathbb{Z}, p \neq -1\) and \(\alpha \in (0, \pi)\). According to Proposition 1.5 in Section I A, we need to discuss the following two sub-cases.

Sub-case A.1 “\(b > 0\)”: each curve \(\Gamma_{p+1}\) is located in between the straight lines \(L_p: y = \alpha + p\pi\) for \(p = 0, 2, 4 \cdots\) and \(p = -3, -5 \cdots\).

When \(A \geq 0\), we have \(1 \geq A/(be^q) \geq 0\) and \(\arccos \frac{A}{be^q} \in (0, \pi/2]\); conversely, when \(A < 0\), we have \(-1 \leq A/(be^q) < 0\) and \(\arccos \frac{A}{be^q} \in (\pi/2, \pi]\). For \(y > 0\), Eq. (S12) becomes

\[
\begin{cases}
    y = \sqrt{b^2e^{2A} - A^2}, \\
    y = \alpha + \arccos \left(\frac{A}{be^q}\right) + 2(q - 1)\pi, \quad q = 1, 2, 3 \cdots,
\end{cases}
\]

(S13)

where the second equation follows from the fact that not \(\alpha - \arccos \left(\frac{A}{be^q}\right) + 2(q - 1)\pi\) but \(\alpha + \arccos \left(\frac{A}{be^q}\right) + 2(q - 1)\pi\) locates in the interval \((\alpha + p\pi, \alpha + (p + 1)\pi)\) for \(p = 0, 2, 4 \cdots\).

In what follows, we regard Eq. (S13) as a group of equations with respect to \(b\). Thereby, we denote the root of Eq. (S13) as a function of \(q\) by \(b_{11}(q)\). Fig. S-5(a) numerically shows the root for particularly chosen \(q, \alpha\) and \(A\). With these settings, we approach the following conclusion.

**FIG. S-5.** The transverse branches \(\Gamma_1\) (a) and \(\Gamma_2\) (b). Their intersection points with the vertical line \(x = -A\) produce positive values, denoted, respectively, by \(b_{11}(1)\) and \(b_{12}(-2)\). Any root of Eq. (S4) on branch \(\Gamma_1\) (\(\Gamma_2\), resp.) satisfies \(x < -A\) if \(b \in (0, b_{11}(1))\) \([b \in (0, b_{12}(-2)),\) resp.\]. Here, the parameters are \(\alpha = 1\) and \(A = 1.5\).

**Lemma II.3** The function \(b_{11}(q)\) is monotonically increasing with respect to \(q = 1, 2, 3, \cdots\).
proof. In order to show the monotonicity, we investigate the quantity
\[
\sqrt{b_{11}^2(q + 1)e^{2A} - A^2} - \sqrt{b_{11}^2(q)e^{2A} - A^2} = \alpha + \arccos \frac{A}{b_{11}(q + 1)e^A} + 2q\pi - \left[ \alpha + \arccos \frac{A}{b_{11}(q)e^A} + 2(q - 1)\pi \right]
\]
\[
= 2\pi + \left[ \arccos \frac{A}{b_{11}(q + 1)e^A} - \arccos \frac{A}{b_{11}(q)e^A} \right],
\]
where the last term is always positive because the range of \( \frac{A}{be^A} \) is \([0, \pi]\) as mentioned above. Therefore, \( b_{11}(q) < b_{11}(q + 1) \) for all \( q = 1, 2, 3, \cdots \), which completes the proof. ■

Now, according to Proposition I.5, all the roots of Eq. (S4) on the branches \( \Gamma_{p+1} \) for \( p = 0, 2, 4 \cdots \) satisfy \( x < -A \) if
\[
b \in \bigcap_{q=1}^{\infty} (0, b_{11}(q)) = (0, b_{11}(1)).
\]
Here, the last equality follows from Lemma II.3, and \( b_{11}(1) \) satisfies the equation:
\[
\sqrt{b_{11}^2(1)e^{2A} - A^2} = \alpha + \arccos \frac{A}{b_{11}(1)e^A}.
\]
Moreover, suppose \( b_{12}(q) \) to be the root of the following equation:
\[
\begin{align*}
y &= -\sqrt{b_{12}^2(q)e^{2A} - A^2}, \\
y &= \alpha - \arccos \frac{A}{be^A} + 2(q + 1)\pi, \quad q = -2, -3 \cdots,
\end{align*}
\]
which follows directly from Eq. (S12) for \( y < 0 \). Fig. S-5(b) shows this root for particularly selected parameters. Analogously, we can verify that the function \( b_{12}(q) \) is monotonically decreasing for \( q = -2, -3, \cdots \). Hence,
\[
\bigcap_{q=-2}^{-\infty} (0, b_{12}(q)) = (0, b_{12}(-2)),
\]
which, together with Proposition I.5, implies that all the roots of Eq. (S4) on the branches \( \Gamma_{p+1} \) for \( p = -3, -5 \cdots \) satisfy \( x \leq A \), provided with \( b \in (0, b_{12}(-2)) \). Here, \( b_{12}(-2) \) satisfies the equation
\[
\sqrt{b_{12}^2(-2)e^{2A} - A^2} = -\alpha + \arccos \frac{A}{b_{12}(-2)e^A} + 2\pi.
\]
Next, we compare the values of \( b_{12}(-2) \) and \( b_{11}(1) \). To this end, we investigate the quantity
\[
\sqrt{b_{12}^2(-2)e^{2A} - A^2} - \sqrt{b_{11}^2(1)e^{2A} - A^2} = -\alpha + \arccos \frac{A}{b_{12}(-2)e^A} + 2\pi - \alpha - \arccos \frac{A}{b_{11}(1)e^A}
\]
\[
= \left[ \arccos \frac{A}{b_{12}(-2)e^A} - \arccos \frac{A}{b_{11}(1)e^A} \right] + 2(\pi - \alpha),
\]
which is absolutely non-negative for any \( \alpha \in (0, \pi/2] \). Hence, \( b_{12}(-2) \geq b_{11}(1) \) if \( \alpha \in (0, \pi/2] \).

Altogether, on the branches \( \Gamma_{p+1} \) for \( p = 0, 2, 4 \cdots \) and \( p = -3, -5 \cdots \), all the corresponding roots of Eq. (S4) satisfy \( x < -A \) if and only if either one of the following conditions is satisfied:

(I-1) \( \alpha \in (0, \pi/2] \) and \( b \in (0, b_{12}(1)) \);
(I-2) \( \alpha \in (\pi/2, \pi] \) and \( b \in (0, \min(b_{11}(1), b_{12}(-2))) \).

Sub-case A.II “\( b < 0 \)”:

each curve \( \Gamma_{p+1} \) is located in between the straight lines \( L_{p+1} \) and \( L_p \) for \( p = 1, 3, 5 \cdots \) and \( p = -2, -4, -6 \cdots \).

Clearly, when \( A \leq 0 \), we have \( 0 \leq A/(be^A) \leq 1 \) and \( \arccos \frac{A}{be^A} \in [0, \pi/2] \); conversely, when \( A > 0 \), we have \( 0 > A/(be^A) \geq 1 \) and \( \arccos \frac{A}{be^A} \in (\pi/2, \pi] \). Suppose \( b_{21}(q) \) and \( b_{22}(q) \) to be, respectively, the roots of the equations:
\[
\begin{align*}
y &= \sqrt{b_{21}^2(q)e^{2A} - A^2} > 0, \\
y &= \alpha - \arccos \frac{A}{be^A} + 2(q - 1)\pi, \quad q = 2, 3 \cdots
\end{align*}
\]
and

\[
\begin{align*}
  y &= -\sqrt{b^2 e^{2A} - A^2} < 0, \\
  y &= \alpha + \arccos \frac{A}{be^A} + 2q\pi, \quad q = -1, -2, -3 \cdots,
\end{align*}
\]

where the above two equations follow from Eq. (S12) for \( y > 0 \) and \( y < 0 \), respectively. Figures S-6(a) & S-6(b) show the roots \( b_{21}(q) \) and \( b_{22}(q) \) when parameters are particularly selected.

**FIG. S-6.** The transverse branches \( \Gamma_2 \) (a) and \( \Gamma_{-1} \) (b). Their intersection points with the vertical line \( x = -A \) produce negative values, denoted, respectively, by \( b_{21}(2) \) and \( b_{22}(-1) \). Any root of Eq. (S4) on branch \( \Gamma_2 \) (\( \Gamma_{-1} \), resp.) satisfies \( x < -A \) if \( b \in (b_{21}(2), 0) \) [\( b \in (b_{22}(-1), 0) \), resp.]. Here, the parameters are \( \alpha = 1 \) and \( A = 1.5 \).

Direct computations yield that \( b_{21}(q) \) is strictly decreasing for \( q = 2, 3, \cdots \), and that \( b_{22}(q) \) is strictly increasing for \( q = -1, -2, -3, \cdots \). Therefore, we have

\[
\bigcap_{q = 2}^{+\infty} (b_{21}(q), 0) = (b_{21}(2), 0), \quad \bigcap_{q = -1}^{-\infty} (b_{22}(q), 0) = (b_{22}(-1), 0),
\]

where \( b_{21}(2) \) and \( b_{22}(-1) \) satisfies, respectively, the equations:

\[
\sqrt{b_{21}^2(2)e^{2A} - A^2} = \alpha - \arccos \frac{A}{b_{21}(2)e^A} + 2\pi, \quad \sqrt{b_{22}^2(-1)e^{2A} - A^2} = -\alpha - \arccos \frac{A}{b_{22}(-1)e^A} + 2\pi.
\]

In particular, we have \( b_{21}(2) \leq b_{22}(-1) \) for \( \alpha \in [\pi/2, \pi] \) because of \( b < 0 \) and because

\[
\sqrt{b_{21}^2(2)e^{2A} - A^2} - \sqrt{b_{22}^2(-1)e^{2A} - A^2} = \alpha - \arccos \frac{A}{b_{21}(2)e^A} + 2\pi - \left[ -\alpha - \arccos \frac{A}{b_{22}(-1)e^A} + 2\pi \right]
\]

\[
= 2\alpha - \left[ \arccos \frac{A}{b_{21}(2)e^A} - \arccos \frac{A}{b_{22}(-1)e^A} \right]
\]

is always non-negative for any \( \alpha \in [\pi/2, \pi] \).

Consequently, on the branches \( \Gamma_{p+1} \) for \( p = 1, 3, 5 \cdots \) and \( p = -2, -4, -6 \cdots \), all the corresponding roots of Eq. (S4) satisfy \( x < -A \) if and only if either one of the following conditions is satisfied:

- (II-1) \( \alpha \in (0, \pi/2) \) and \( b \in (\max(b_{21}(2), b_{22}(-1)), 0) \);
- (II-2) \( \alpha \in [\pi/2, \pi] \) and \( b \in (b_{22}(-1), 0) \).

**B. Along the concave branch for non-real-valued \( \beta \)**

Secondly, we consider the branch \( \Gamma_0 \), which is obtained in Section I A. As shown in Fig. S-1, \( \Gamma_0 \) is located in between the asymptotes \( L_0 : y = \alpha \) and \( L_{-1} : y = \alpha - \pi \). The upper half of \( \Gamma_0 \) corresponds to the case of \( b < 0 \), while the lower half to the case of \( b > 0 \). \( b = 0 \) corresponds to the origin \( (x, y) = (0, 0) \), which has been excluded from the following discussion. As illustrated in Eq. (S7), the leftmost vertex of \( \Gamma_0 \) is \((-\cos^2(y' - \alpha), y')\) where
$y = y^*(a)$ is the unique solution of $2y = \sin(2(y - \alpha))$, and the locus of the vertex is on a circle of radius $1/2$ and center $(-1/2, 0)$, passing through the origin when $\alpha = \pi/2$.

First, $-\cos^2(y^* - \alpha) < -A$ is required in the following discussion; otherwise no part of the concave branch $\Gamma_0$ locates on the left side of the vertical line $x = -A$, which implies that the root of Eq. (S4) on $\Gamma_0$ satisfies $x \geq -A$ and then the asymptotical stability of the controlled system (S1) cannot be assured.

As mentioned above, the $y$-axis of the root on $\Gamma_0$ is allowed to be either positive or negative. Thus, from Eq. (S12), it follows that

$$\begin{cases} y = \pm \sqrt{b^2e^{2A} - A^2}, \\ y = \alpha - \arccos \frac{A}{be^A} \in (\alpha - \pi, \alpha). \end{cases} \tag{S14}$$

Moreover, according to Lemma I.2, we know that there are two interaction points between $\Gamma_0$ and the $y$-axis. One is the origin and the other is $(0, y_0)$ with $y_0 = \alpha - \pi/2$. Thus, for the situation of $a \in (0, \pi/2)$, the vertex of $\Gamma_0$ lies in the lower half plane (i.e., $y < 0$). For this situation, we divide the following discussions into three sub-cases.

**Sub-case B.I 
$0 < A < \cos^2(y^* - \alpha)$**: The concave branch $\Gamma_0$ intersects with the line $x = -A$ at the two points which are located on the left side of the $y$-axis. These two points actually correspond to the two values of $b$, denoted by $b_{01}$ and $b_{02}$. From Proposition I.5 and Eq. (S14), it follows that $b_{01}$ and $b_{02}$ are positive but different-valued, satisfying the equation:

$$\sqrt{b^2e^{2A} - A^2} = -\alpha + \arccos \frac{A}{be^A} \tag{S15}$$

with respect to $b$ for $a \in (0, \pi/2)$. Assume that $b_{01} < b_{02}$, as shown in Fig. 7(a). Then, according to Proposition I.5, the root of Eq. (S4) on $\Gamma_0$ satisfies $x < -A$ if $b \in (b_{01}, b_{02})$.

Note that when $b = b_{01,02}$, we have $A/be^A > 0$, so that $\arccos \frac{A}{be^A} \in (0, \pi/2)$, which is the same as the range of $\alpha$ in the present situation. Thus, from Eq. (S15), we have $\alpha \leq \arccos \frac{A}{be^A}$ as $b = b_{01,02}$. However, it is necessary to analytically show the existence of two different roots of Eq. (S15) for the present sub-case.

![FIG. S-7. The intersecting points between the concave branch $\Gamma_0$ and the vertical line $x = -A$ determine the values of $b_{01,02} (a)$ and $b_{03,04} (b)$, when $a \in (0, \pi/2)$. Here, $b_{01,02,04} > 0$ and $b_{03} < 0$. As $0 < A < \cos^2(y^* - \alpha) (A < 0)$, Eq. (S4) on $\Gamma_0$ satisfies $x < -A$ if $b \in (b_{01}, b_{02}) [b \in (b_{03}, b_{04})]$. Here, the parameters are $a = 1 \in (0, \pi/2)$, $A = 0.025 (a)$, and $A = -0.025 (b)$.](image)

**Lemma II.4** Assume that $a \in (0, \pi/2)$ is a given number and set $A^* = \cos^2(y^* - \alpha)$. Here, $y^*$ is the unique solution of $2y = \sin(2(y - \alpha))$ as obtained in Lemma I.1. For $A \in (0, A^*)$, Eq. (S15) with respect to $b \in \left[\frac{A}{e^A}, +\infty\right)$ has two positive but different roots, denoted by $b_1^A$ and $b_\lambda^A$. In particular, with the increase of $A$ from $0$ to $A^*$, $b_1^A$ increases from $A/e^A$ to $\sqrt[A^*]{e^A}$/e$^A$; while $b_\lambda^A$ decreases from $+\infty$ to $\sqrt[A^*]{e^A}$. Both roots merge into $\sqrt[A^*]{e^A}$ as $A = A^*$. For $A > A^*$, Eq. (S15) has no root.
proof. As verified in Lemma I.1, the root \( y = y^* \) of \( 2y = \sin 2(y - \alpha) \) is uniquely existent. Use the notation \( \eta(y) \) defined in Lemma I.1 and notice that

\[
\begin{align*}
\eta(0) &= -\sin(-2\alpha) = \sin 2\alpha > 0, \\
\eta(-\pi/2) &= -\pi - \sin[2(-\pi/2 - \alpha)] = -\pi - \sin 2\alpha < 0, \\
\eta'(y) &= 2 - 2\cos 2(y - \alpha) \geq 0,
\end{align*}
\]

for any given \( \alpha \in (0, \pi/2) \). Therefore, the root \( y = y^* \in (-\pi/2, 0) \), which gives \( y^* - \alpha \in (-\pi, 0) \) and \( \sin(y^* - \alpha) < 0 \). Notice that

\[
0 > 2y^* = \sin 2(y^* - \alpha) = 2 \sin(y^* - \alpha) \cos(y^* - \alpha),
\]

Then, we obtain

\[
\cos(y^* - \alpha) > 0, \quad y^* - \alpha \in (-\pi/2, 0).
\]

As set in the lemma, \( A^* = \cos^2(y^* - \alpha) \). This with (S16) implies \( 0 < A^* < 1 \). Furthermore, \( \cos(y^* - \alpha) = \sqrt{A^*} \), which yields \( y^* = \alpha - \arccos \sqrt{A^*} \).

Denote by \( u(b, A) \equiv u_1(b, A) - u_2(b, A) \) where

\[
u_1(b, A) = \sqrt{b^2e^{2A} - A^2}, \quad u_2(b, A) = -\alpha + \arccos \frac{A}{be^A},
\]

and \( b \geq A/e^A > 0 \) because of the condition (C-I) as assumed above. Thus, the derivative of \( u(b, A) \) with respect to \( b \) is

\[
u_0(b, A) = \frac{\partial u(b, A)}{\partial b} = \frac{b^2 e^{2A} - A}{b \sqrt{b^2 e^{2A} - A^2}},
\]

so that \( u_0(b, A) = 0 \) gives a critical value \( b = \hat{b}(A) \equiv \sqrt{A}/e^A \). Moreover, \( u_0(b, A) = 0 \) coupled with \( u(b, A) = 0 \) yields an equation with respect to \( A \):

\[
\sqrt{A(1 - A)} = -\alpha + \arccos \sqrt{A}.
\]

By using the results obtained in (S16), one can verify directly that \( A = A^* = \cos^2(y^* - \alpha) \) is a root of Eq. (S19). Hence, \( u(\hat{b}(A^*), A^*) = 0 \).

Now, we consider the case of \( 0 < A < A^* < 1 \). Then, \( A/e^A < \hat{b}(A) \), and the derivative \( u_0(b, A) \) is negative as \( b \in (A/e^A, \hat{b}(A)) \) but positive as \( b \in (\hat{b}(A), +\infty) \). In particular, \( u(A/e^A, A) = \alpha > 0 \) and \( u(b, A) \to +\infty \) as \( b \to +\infty \); while

\[
u(\hat{b}(A), A) = \sqrt{A(1 - A)} + \alpha - \arccos \sqrt{A} < 0,
\]

for \( 0 < A < A^* \), because \( u(\hat{b}(A^*), A^*) = 0 \) as verified above, and because

\[
\frac{du(\hat{b}(A), A)}{dA} = \frac{1 - A}{\sqrt{A - A^2}} > 0,
\]

for \( 0 < A < A^* < 1 \). Therefore, from the continuity of the function \( u(b, A) \) with respect to the argument \( b \), it follows that there are two positive zeros \( b_1^{A} \) and \( b_2^{A} \) of \( u(b, A) \), respectively, in the intervals \( (A/e^A, \hat{b}(A)) \) and \( (\hat{b}(A), +\infty) \) if \( 0 < A < A^* \).

As a matter of fact, notice the derivative

\[
u_4(\hat{b}(A^*), A^*) = \frac{\partial u(b, A)}{\partial A} \bigg|_{b = \hat{b}(A^*), A = A^*} = \frac{1 - A^*}{\sqrt{A^* - (A^*)^2}} \neq 0.
\]

Then, according to the Implicit Function Theorem [2], there exists a smooth function \( A = A(b) \) in the neighborhood of \( b = \hat{b}(A^*) \) such that \( u(A(b), b) = 0 \) and \( A(\hat{b}(A^*)) = A^* \). Moreover,

\[
\frac{du(\hat{b}(A), A)}{dA} = \frac{\hat{b}^2 e^{2A} - 2A(\hat{b}) + 1}{\sqrt{\hat{b}^2 e^{2A} - A(\hat{b})^2}} A'(\hat{b}) + be^{2A\hat{b}} - A(b)/b.
\]
which yields $A'(\hat{b}(A^*)) = 0$ and $A''(\hat{b}(A^*)) = -2e^{2\hat{b}}/(1-A^*) < 0$. This implies that the graph of $A = A(b)$ describes the variation of the two zeros $b^A_{1,2}$ with the change of $A \in (0, A^*)$. Indeed, these two zeros merge into $b = \hat{b}(A^*)$ as $A = A^*$, so $u(b, A)$ has no zero as $A > A^*$.

In fact, we can directly verify $u(b, A)$ has no zero as $A > A^*$. On the one hand, for $A \geq 1 > A^*$, we have $\hat{b}(A) \leq A/e^A$ and $u(b, A) \geq 0$ as $b \in [A/e^A, +\infty)$. Hence, $u(b, A) \geq u(A/e^A) = 0$ as $b \in [A/e^A, +\infty)$. On the other hand, consider $1 > A > A^*$. From the monotonicity of $u(b, A)$ with respect to $b$ as verified above, it follows that $u(b, A)$ approaches the minimal value as $b = \hat{b}(A)$. Note that the minimal value $u(b, A)$ equals to zero as $A = A^*$ and that $u(b(A), A)$ is strictly increasing due to the validity of the inequality in (S20) for $1 > A > A^*$. Consequently, $u(b, A) > 0$ for $1 > A > A^*$ and $b \in [A/e^A, +\infty)$. Finally, we complete the entire proof.

Figures S-8(a)-(c) show the variation of the roots of Eq. (S15) with changing $A$ for particularly selected parameters. As shown in Fig. S-8(c), there are two positive but different-valued roots of Eq. (S15) for some $A \in (0, A^*)$.

Sub-case B.II “$A = 0$”: The concave branch $\Gamma_0$ intersects directly with the $y$-axis at the two points which correspond to the two values of $b$, viz., 0 and $\pi/2 - \alpha > 0$. From Proposition I.5, it follows that the root of Eq. (S4) on $\Gamma_0$ satisfies $x < -A$ if $b \in (0, \pi/2 - \alpha)$.

Sub-case B.III “$A < 0$”: The concave branch $\Gamma_0$ intersects with the line $x = -A$ at the two points which are located on the right side of the $y$-axis, as shown in Fig. 7(b). These two points correspond to the two values of $b$, denoted by $b_{03}$ and $b_{04}$. From Proposition I.5 and Eq. (S14), it follows that $b_{03} < 0$ satisfies the equation:

$$\sqrt{b^2e^{2A}} - A^2 = -\arccos \frac{A}{be^A},$$

(S21)

with respect to $b$ for $\alpha \in (0, \pi/2)$. Meanwhile, $b_{04} > 0$ satisfies Eq. (S15) with respect to $b$ for $\alpha \in (0, \pi/2)$. Then, according to Proposition I.5, the root of Eq. (S4) on the branch $\Gamma_0$ satisfies $x < -A$ if $b \in (b_{03}, b_{04})$.

Also note that $\arccos \frac{A}{be^A} \in (0, \pi/2)$ as $A < 0$ and $b_{03} < 0$; while $\arccos \frac{A}{be^A} \in (\pi/2, \pi)$ as $A < 0$ and $b_{04} > 0$. With these, $A < 0$, and $\alpha \in (0, \pi/2)$, the existence of a unique root for each of Eqs. (S21) and (S15) can be verified as follows.

**Lemma II.5** For any given $\alpha \in (0, \pi/2)$ and $A < 0$, Eq. (S21) with respect to $|b| \geq -A/e^A$ has a unique root, which is negative and denoted by $b^A_{3}$; while, Eq. (S15) with respect to $|b| \geq -A/e^A$ has a unique root, which is positive and denoted by $b^A_{4}$.

**proof.** Consider the function $u(b, A)$ as defined in (17). Clearly, from the derivative (18), it follows that, for $A < 0$, $u(b, A)$ as a function of $b$ is decreasing in $(-\infty, A/e^A)$ but increasing in $(-A/e^A, +\infty)$. Moreover, $u(b, A)$

---

FIG. S-8. The variation of the roots of Eq. (S15) with the change of $A$. Here, the parameters are taken as: $\alpha = 1 \in (0, \pi/2)$, $A = 0.1$ (a), $A = 0.0838$ (b), and $A = 0.05$ (c), so that $A^* = \cos^2(\gamma - \alpha) = 0.0838$. 

![Diagram](image-url)
we can verify that \( b_4 = -\frac{\sqrt{A}}{e^A} + A \). Correspondingly, the vertex of the concave branch \( u \). Analogous to Sub-case B.I., \( u \), \( w \), and \( v \), we write \( u(b, A) = u_2(b, A) \), where \( |b| \geq -A/e^A, A < 0 \), and \( u_{1,2}(b, A) \) are defined in (S17). Thus, the derivative of \( u(b, A) \) with respect to \( b \) can be computed as:

\[
\frac{\partial u(b, A)}{\partial b} = \frac{b^2 e^{2A} + A}{b \sqrt{b^2 e^{2A} - A^2}}.
\]

For the case of \( A \leq -1 \), the derivative in (S22) implies that \( u(b, A) \) as a function \( b \) is decreasing in \((-\infty, A/e^A)\) but increasing in \((-A/e^A, +\infty)\). Note that \( u'(b, A) \to +\infty \) as \( b \to +\infty \). Thus, the derivative of \( u(b, A) \) at the two roots \( b_3 = -\frac{\sqrt{A}}{e^A} \) and \( b_4 = -\frac{\sqrt{A}}{e^A} + A \) satisfies \( u'(b_3) = u'(b_4) = 0 \), i.e., \( A < 0 \). Therefore, from the continuity and monotonicity of \( u(b, A) \), it follows that Eq. (S15) has only one negative root \( b_3 < 0 \). Moreover, for the case of \(-1 < A < 0\), the derivative in (S22) implies that \( u(b, A) \) as a function \( b \) is decreasing, respectively, in \((-\infty, -\sqrt{A}/e^A)\) and \((-A/e^A, \sqrt{A}/e^A)\) but increasing, respectively, in \((-\sqrt{A}/e^A, A/e^A)\) and \((\sqrt{A}/e^A, +\infty)\). Also note that \( u(b, A) \to +\infty \) as \( b \to -\infty \) and \( u(-\sqrt{A}/e^A, A) = -A < 0 \). Therefore, from the continuity and monotonicity of \( u(b, A) \), it follows that Eq. (S15) has only one negative root \( b_3 < 0 \) and no root in \((-\sqrt{A}/e^A, A/e^A)\). In addition, \( u(-\sqrt{A}/e^A, A) = -A + \sqrt{A}/e^A \). Note that the derivative \( u' = (-1 - A)/\sqrt{A - A^2} < 0 \) for \(-1 < A < 0 \), and that \( u(0) = -A + \pi/2 > 0 \). Hence, \( u(-\sqrt{A}/e^A, A) = -A + \sqrt{A}/e^A \). Consequently, Eq. (S15) has no root in the intervals \((-\sqrt{A}/e^A, \sqrt{A}/e^A)\) and \((\sqrt{A}/e^A, +\infty)\).

Next, we deal with the situation of \( \alpha \in (\pi/2, \pi) \). Correspondingly, the vertex of the concave branch \( \Gamma_0 \) lies in the upper half plane (i.e., \( y > 0 \)). Still, we have three sub-cases pending for discussion.

**FIG. S-9.** The intersecting points between the concave branch \( \Gamma_0 \) and the line \( x = -A \) determine the values of \( b_{05,06} \) (a) and \( b_{05,08} \) (b), when \( \alpha \in (\pi/2, \pi) \). Here, \( b_{05,06,07} < 0 \) and \( b_{06} > 0 \). As \( 0 < A \leq \cos^2(\pi/2 - \alpha) \), the root of Eq. (S4) on \( \Gamma_0 \) satisfies \( x < -A \). If \( b \in (b_{05}, b_{06}) \), \( |b| < b_{05} \). For \( b \in (b_{06}, b_{08}) \), \( |b| > b_{06} \). All the parameters are \( \alpha = 2 \in (\pi/2, \pi) \), \( A = 0.025 \), and \( A = -0.025 \).}

**Sub-case B.IV \( 0 < A < \cos^2(\pi/2 - \alpha) \):** The concave branch \( \Gamma_0 \) intersects with the line \( x = -A \) at the two points which are located on the left side of the \( y \)-axis, which is analogous to Sub-case B.I. As shown in Fig. 9(a), these two points actually correspond to the two values of \( b \), denoted by \( b_{05} \) and \( b_{06} \). Using the method similar to the proof performed for Lemma II.4, we can verify that \( b_{05} \) and \( b_{06} \) are negative but different-valued, satisfying Eq. (S21) with respect to \( b \) for \( \alpha \in (\pi/2, \pi) \). When \( b = b_{05,06} \), we have \( A/e^A \). Hence, \( A < 0 \), such that \( \arccos \frac{b}{e^A} \in (\pi/2, \pi) \). Assume \( b_{05} < b_{06} \). Then, according to Proposition I.5, the root of Eq. (S4) on \( \Gamma_0 \) satisfies \( x < -A \). If \( b \in (b_{05}, b_{06}) \).

**Sub-case B.V \( A = 0 \):** Analogous to Sub-case B.II, \( \Gamma_0 \) intersects with the \( y \)-axis at the two points which correspond to the two values of \( b \), viz., 0 and \( \pi/2 - \alpha < 0 \). From Proposition I.5, it follows that the root of Eq. (S4) on \( \Gamma_0 \) satisfies \( x < -A \).
Sub-case B.VI “$A < 0$”: The concave branch $\Gamma_0$ intersects with the line $x = -A$ at the two points which are located on the right side of the $y$-axis, which is analogous to Sub-case B. III. As shown in Fig. 9(b), these two points correspond to the two values of $b$, denoted by $b_{07}$ and $b_{08}$. From Proposition I.5 and Eq. (S14), it follows that $b_{07} < 0$ and $b_{08} > 0$ satisfy, respectively, Eqs. (S21) & (S15) for $\alpha \in (\pi/2, \pi)$. The unique root existence of each of these equations can be verified through a method analogous to the proof of Lemma II.5. Also it is noted that $\arccos \frac{A}{b_{07}e^A} \in (0, \pi/2)$ as $A < 0$ and $b_{07} < 0$; while $\arccos \frac{A}{b_{08}e^A} \in (\pi/2, \pi)$ as $A < 0$ and $b_{08} > 0$.

Finally, according to Proposition I.5, the root of Eq. (S4) on the branch $\Gamma_0$ satisfies $x < -A$ if $b \in (b_{07}, b_{08})$.

Thirdly, we cope with the critical situation of $\alpha = \pi/2$. Thus, the vertex of the branch $\Gamma_0$ is located at the origin $(0, 0)$, so we only need to consider the case of $A < 0$. As shown in Fig. S-10, the concave branch $\Gamma_0$ intersects with the line $x = -A$ at the two points which correspond to the two values of $b$, denoted by $b_{09}$ and $b_{10}$. From Proposition I.5 and Eq. (S14), it follows that $b_{09} < 0$ and $b_{10} > 0$ satisfy, respectively, Eqs. (S21) and (S15) for $\alpha = \pi/2$. Consequently, according to Proposition I.5, the root of Eq. (S4) on the branch $\Gamma_0$ satisfies $x < -A$ if $b \in (b_{09}, b_{10})$. This conclusion actually could be included in the Sub-case B.VI.

![Fig. S-10](image-url)  

In order to summarize all the conclusions obtained in the above Subsections II-A & II-B and provide compact conditions for stability, we first give results on the relations among the ending points of the intervals for $b$.

**Lemma II.6** For any given $\alpha \in (0, \pi/2)$, max $\{b_{03}, b_{21}(2), b_{22}(-1)\} = b_{03}$.

**proof.** First, the function $u_1(b, A)$ as defined in (S17) is decreasing with respect to $b$ in $L^- \doteq (-\infty, -|A|/e^A)$ and increasing in $L^+ \doteq ([|A|/e^A, +\infty)$ due to the unchangeable sign of the derivative

$$\frac{\partial u_1(b, A)}{\partial b} = \frac{be^{2A}}{\sqrt{b/e^A - A^2}}$$

as $b$ belongs to either one of the intervals $L^\pm$. With this monotonicity, we are able to validate the conclusion of the lemma.

To this end, we investigate the following two quantities:

$$u_1(b_{21}(2), A) - u_1(b_{03}, A) = \alpha - \arccos \frac{A}{b_{21}(2)e^A} + 2\pi - \left[\alpha - \arccos \frac{A}{b_{03}e^A}\right]$$

$$= 2\pi - \left[\arccos \frac{A}{b_{21}(2)e^A} - \arccos \frac{A}{b_{03}e^A}\right] > 0$$

and

$$u_1(b_{22}(-1), A) - u_1(b_{03}, A) = -\alpha - \arccos \frac{A}{b_{22}(-1)e^A} + 2\pi - \left[\alpha - \arccos \frac{A}{b_{03}e^A}\right]$$

$$= 2(\pi - \alpha) - \left[\arccos \frac{A}{b_{22}(-1)e^A} - \arccos \frac{A}{b_{03}e^A}\right] > 0,$$
where the validity of the two inequalities is due to \( \arccos \frac{1}{e^4} \in [0, \pi] \) and \( \alpha \in (0, \pi/2) \). Note that \( b_{03}, b_{21}(2), \) and \( b_{22}(-1) \) are all negative, which have been verified in the preceding argument. Then, from the monotonicity of the function \( u_1(b, A) \), it follows that \( b_{03} > b_{21}(2) \) and \( b_{03} > b_{22}(-1) \). This therefore validates \( \max\{b_{03}, b_{21}(2), b_{22}(-1)\} = b_{03} \).

**Lemma II.7** For any given \( \alpha \in (\pi/2, \pi) \), \( \min\{b_{08}, b_{11}(1), b_{12}(-2)\} = b_{08} \).

**proof.** We investigate the following two quantities:

\[
\begin{align*}
\frac{d}{dA} \arccos \left( \frac{A}{(1)e^4} \right) &= \frac{-\frac{A}{(1)e^4}}{\sqrt{1 - \left( \frac{A}{(1)e^4} \right)^2}}, \\
\frac{d}{dA} \arccos \left( \frac{A}{b_{08}e^4} \right) &= \frac{-\frac{A}{b_{08}e^4}}{\sqrt{1 - \left( \frac{A}{b_{08}e^4} \right)^2}},
\end{align*}
\]

and

\[
\begin{align*}
\arccos \left( \frac{A}{b_{08}e^4} \right) - \arccos \left( \frac{A}{(1)e^4} \right) &= \frac{2\pi}{A} + \frac{-\frac{2\pi}{b_{08}e^4}}{\sqrt{1 - \left( \frac{A}{b_{08}e^4} \right)^2}} - \frac{-\frac{2\pi}{(1)e^4}}{\sqrt{1 - \left( \frac{A}{(1)e^4} \right)^2}} < 0,
\end{align*}
\]

where the validity of the two inequalities follows from \( \arccos \frac{1}{e^4} \in [0, \pi] \) and the assumed condition \( \alpha \in (\pi/2, \pi) \). Thus, the monotonicity of \( u_1(b, A) \), together with the fact that \( b_{11}(1), b_{12}(-2), \) and \( b_{08} \) are all positive, implies \( b_{11}(1) > b_{08} \) and \( b_{12}(-2) > b_{08} \). This therefore completes the proof.

**Lemma II.8** For any given \( \alpha \in (0, \pi/2) \), \( \max\{b_{11}(1), b_{02}\} = b_{11}(1) \) if \( 0 < A < 1 \); while \( \max\{b_{11}(1), b_{04}\} = b_{11}(1) \) if \( A < 0 \).

**proof.** For the case of \( 0 < A < 1 \), we consider the function

\[
u_3(b) = \sqrt{b^2e^{2A} - A^2} - \arccos \frac{A}{be^4},\]

with \(|b| \geq |A|/e^4 > 0\). From the derivative \( u_3'(b) \) which is the same as the derivative in (S18), it follows that \( u_3(b) \) is decreasing in \( (A/e^4, \sqrt{A}/e^4) \) but increasing in \( (\sqrt{A}/e^4, +\infty) \).

Moreover, \( u_3(A/e^4) = 0 \) and \( u_3(\sqrt{A}/e^4) = \sqrt{A(1 - A)} - \arccos \sqrt{A} \). The derivative

\[
\frac{du_3(\sqrt{A}/e^4)}{dA} = \frac{1 - A}{\sqrt{A - A^2}} > 0,
\]

as \( A \in (0, 1) \). Since \( u_3(A/e^4)|_{A=1} = 0 \), \( u_3(\sqrt{A}/e^4) \) is negative as \( A \in (0, 1) \). This, together with the monotonicity of \( u_3(b) \) and

\[
0 < \alpha = u_3(b_{11}(1)) > u_3(b_{02}) = -\alpha < 0,
\]

implies that \( b_{11}(1) > b_{02} \).

As for the case of \( A < 0 \), \( u_3(b) \) is increasing in \( (-A/e^4, +\infty) \) and the inequalities in (S23) are still valid. Therefore, we conclude \( b_{11}(1) > b_{04} \).

**Lemma II.9** For any given \( \alpha \in (\pi/2, \pi) \), \( \min\{b_{22}(-1), b_{03}\} = b_{22}(-1) \) if \( 0 < A < 1 \); while \( \min\{b_{22}(-1), b_{07}\} = b_{22}(-1) \) if \( A < 0 \).

**proof.** For the case of \( 0 < A < 1 \), we consider the function

\[
u_4(b) = \sqrt{b^2e^{2A} - A^2} + \arccos \frac{A}{be^4},\]

with \(|b| \geq |A|/e^4 > 0\). From the derivative \( u_4'(b) \) which is the same as the derivative in (S22), it follows that \( u_4(b) \) is decreasing in \( (-\infty, -A/e^4) \). Note that both \( b_{22}(-1) \) and \( b_{08} \) are less than \(-A/e^4 \), and that \( u_4(b_{22}(-1)) = \]
$2\pi - \alpha > \alpha = u_4(b_{05})$ because of $\alpha \in (\pi/2, \pi)$. Therefore, from the monotonicity of $u_4(b)$, it follows that $b_{22}(-1) < b_{05}$.

As for the case of $A < 0$, we are to consider two situations. On the one hand, if $A \leq -1$, $u_4(b)$ is decreasing in $(-\infty, A/e^4)$. This, with $u_4(b_{22}(-1)) = 2\pi - \alpha > \alpha = u_4(b_{07})$, implies that $b_{22}(-1) < b_{07}$.

On the other hand, if $-1 < A < 0$, $u_4(b)$ is decreasing in $(-\infty, -\sqrt{-A}/e^4)$ but increasing in $(-\sqrt{-A}/e^4, A/e^4)$.

Moreover, $u_4(A/e^4) = 0$ and $u_4(-\sqrt{-A}/e^4) = \sqrt{-A(1 + A)} - \arccos \sqrt{-A}$ is negative as $A \in (-1, 0)$ because the derivative

$$\frac{du_4(-\sqrt{-A}/e^4)}{dA} = \frac{-1 - A}{\sqrt{-A - A^2}} < 0,$$

for $A \in (-1, 0)$ and $u_4(-\sqrt{-A}/e^4)|_{A=-1} = 0$. Hence, both $b_{22}(-1)$ and $b_{07} \in (-\infty, -\sqrt{-A}/e^4)$. Consequently, from $u_4(b_{22}(-1)) = 2\pi - \alpha > \alpha = u_4(b_{07}) > 0$, it follows that $b_{22}(-1) < b_{07}$.

Altogether, we are able to summarize all the conclusions above, and finally approach a conclusion that, for any non-real-valued $\beta$, every root of Eq. (S4) on the curve $\Gamma$ satisfies $x < -A$ if and only if one of the following conditions is satisfied:

1. $\alpha \in (0, \pi/2), 0 < A < \cos^2(y^* - \alpha)$, and $b \in (b_{01}, b_{02})$;
2. $\alpha \in (0, \pi/2), A = 0$, and $b \in (0, \pi/2 - \alpha)$;
3. $\alpha \in (0, \pi/2), A < 0$, and $b \in (b_{03}, b_{04})$;
4. $\alpha \in (\pi/2, \pi), 0 < A < \cos^2(y^* - \alpha)$, and $b \in (b_{05}, b_{06})$;
5. $\alpha \in (\pi/2, \pi), A = 0$, and $b \in (\pi/2 - \alpha, 0)$;
6. $\alpha \in [\pi/2, \pi), A < 0$, and $b \in (b_{07}, b_{08})$.

### C. Along the branches for real-valued $\beta$

Finally, we consider the case of real-valued $\beta$ (i.e., $\alpha = 0$), in which the elementary root distribution of Eq. (S4) has been discussed in Section 1B.

By an argument similar to those performed in Section II A and by using Proposition 1.7 and Eq. (S12), we can show that on every branch $\Gamma_p$ for $p \in \mathbb{Z}$ with $p \neq -1$, the corresponding root of Eq. (S4) satisfies $x < -A$ if $b \in (b_{01}(1), b_{02}(2))$. Here, $b_{01}(1)$ is negative and $b_{02}(2)$ is positive, as shown in Fig. S-11. They satisfy, respectively, the equations:

$$\sqrt{b_0^2(j)e^{2A} - A^2} = \arccos \frac{A}{b_0(j)e^4} + j\pi,$$

for $j = 1, 2$. Secondly, still according to Proposition 1.7 and Eq. (S12), the concave branch $\Gamma_0$ surely passes through the point $(-1, 0)$. In order to establish the stability condition, we have to locate the point $(-1, 0)$ on the left side of the line $x = -A$, which implies a prerequisite $A < 1$. Then, we easily approach a conclusion that on the branch $\Gamma_0$ and $x$-axis, all the corresponding roots of Eq. (S4) satisfy $x < -A$ if $A < 1$ and $b \in (A/e^4, b_0(0))$. Here, $b_0(0)$ is a positive root satisfying Eq. (S24) for $j = 0$.

Finally, in order to summarize the stability condition for the present case, we perform the computation as follows:

$$u_1(b_0(2), A) - u_1(b_0(0), A) = \arccos \frac{A}{b_0(2)e^4} + 2\pi - \arccos \frac{A}{b_0(0)e^4} = 2\pi + \left[\arccos \frac{A}{b_0(2)e^4} - \arccos \frac{A}{b_0(0)e^4}\right] > 0,$$

where $u_1(b, A)$ is defined in (S17). Hence, $b_0(2) > b_0(0)$. Moreover, $b_0(1) \leq A/e^4$ due to the domain of the function $u_1(b, A)$. Consequently, for the case of real-valued $\beta$ (i.e., $\alpha = 0$), all the corresponding roots of Eq. (S4) satisfy $x < -A$ if and only if $A < 1$ and $b \in (A/e^4, b_0(0))$.

Now, we can summarize all the condition established in the preceding subsections into the NS condition for the asymptotical stability of the controlled system (S1). In the statement of the following theorem, we substitute the original variables of the controlled system (S1) into the conditions obtained above.

**Theorem II.10** The asymptotical stability of the controlled system (S1) is assured if and only if one of the following conditions is satisfied:

1. $\alpha = 0$, $\alpha \tau < 1$, and $-k\tau e^{-\alpha \tau} \in (\alpha e^{-\alpha \tau}, b_0(0))$;
II.  Here, Table I. (I) and setting II. are illustrated as follows:

In addition, for Case III, we need a determination of \( \alpha \) and setting II. to the corresponding situations. For integrity and practical usefulness, we thus include these results also in Table I. Here, Table I is the same as the table listed in Letter.

More concretely, the asymptotical stability of the controlled system (S1) is assured if and only if those conditions in one of the three cases listed in Tab. I are satisfied. In Tab. I, the conditions for the limit situations of \( \tau = 0 \) and \( \tau \to +\infty \) can be directly obtained from the established conditions by letting \( \tau \) go to the corresponding situations. For integrity and practical usefulness, we thus include these results also in Tab. I. Those parameters used in Tab. I are illustrated as follows: \( \alpha' = \alpha_{0,1} \) in Case I are the two solutions satisfying the equation \( \cos[y' - \alpha'] = A \), where \( A = \alpha \tau < 1 \) and \( y = y' - \alpha' \) is the unique solution of \( 2y = \sin 2(y - \alpha') \). For Case I and \( \alpha \in [0, \alpha_0] \), both \( B = B_{01,02} > 0 \) are the only two solutions of the equation \( H_1(B) = -\alpha + H_2(B) \) which follows from Eq. (S15) and setting \( H_1(B) = (B^2 - A^2)^{1/2}, H_2(B) = \arccos(A/B), \) and \( B = be^A \). Correspondingly, for Case I and \( \alpha \in [\alpha_{0,1}], B = B_{01,02} < 0 \) satisfy \( H_1(B) = \alpha - H_2(B) \) which follows from Eq. (S21). In addition, for Case III, \( B = B_{03,07} < 0 \) are the solutions of \( H_1(B) = -\alpha + H_2(B) \), respectively, for \( \alpha \in [0, \pi/2] \) and \( \alpha \in [\pi/2, \pi] \), and \( B = B_{04,08} > 0 \) satisfy \( H_1(B) = -\alpha + H_2(B) \), respectively, for \( \alpha \in [0, \pi/2] \) and \( \alpha \in [\pi/2, \pi] \). The notation \( \Theta \) represents an empty set for the choice of \( k \).

Note that the conditions (2) and (5) in Theorem II.10 need a determination of \( \alpha \) and \( \tau \) for a given \( \alpha \). Sometimes in real applications, one needs to a determination of \( \alpha \) for properly given \( \alpha \) and \( \tau \), since the original system is given in advance but \( \alpha = \psi - d\tau \) (mod \( \pi \)) contains \( \psi \), the phase of the feedback control gain, which can be adjusted for a given system. Thus, we rewrite Theorem II.10 equivalently into a more compact but feasible form in Tab. I. For integrity and practical usefulness, we thus include these results also in Tab. I. Those parameters used in Tab. I are illustrated as follows: \( \alpha' = \alpha_{0,1} \) in Case I are the two solutions satisfying the equation \( \cos[y' - \alpha'] = A \), where \( A = \alpha \tau < 1 \) and \( y = y' - \alpha' \) is the unique solution of \( 2y = \sin 2(y - \alpha') \). For Case I and \( \alpha \in [0, \alpha_0] \), both \( B = B_{01,02} > 0 \) are the only two solutions of the equation \( H_1(B) = -\alpha + H_2(B) \) which follows from Eq. (S15) and setting \( H_1(B) = (B^2 - A^2)^{1/2}, H_2(B) = \arccos(A/B), \) and \( B = be^A \). Correspondingly, for Case I and \( \alpha \in [\alpha_{0,1}], B = B_{01,02} < 0 \) satisfy \( H_1(B) = \alpha - H_2(B) \) which follows from Eq. (S21). In addition, for Case III, \( B = B_{03,07} < 0 \) are the solutions of \( H_1(B) = -\alpha + H_2(B) \), respectively, for \( \alpha \in [0, \pi/2] \) and \( \alpha \in [\pi/2, \pi] \), and \( B = B_{04,08} > 0 \) satisfy \( H_1(B) = -\alpha + H_2(B) \), respectively, for \( \alpha \in [0, \pi/2] \) and \( \alpha \in [\pi/2, \pi] \). The notation \( \Theta \) represents an empty set for the choice of \( k \).
III. APPLICATIONS

As an application of the NS conditions established above, we now consider the following examples: one is the oscillation suppression in the FitzHugh-Nagumo model, and the other is the oscillation death and synchronization in complex networks.

A. Oscillation suppression in the FitzHugh-Nagumo model

First, we consider the FitzHugh-Nagumo model (FHN) which usually is regarded as a simplification of the Hodgkin-Huxley model of spike generation in squid giant axons \([7, 8]\). It thus is used for describing nerve pulses and sometimes for modeling spiral waves in a two-dimensional excitable medium. The typical FHN reads:

\[
\begin{align*}
\dot{v} &= v - \frac{v^3}{3} - w + I, \\
\dot{w} &= \epsilon(v + \zeta - \delta w),
\end{align*}
\]

(S25)

where \(v\) represents the membrane potential, \(w\) is a recovery variable, \(I\) stands for the magnitude of the input stimulus current, and \(\epsilon, \zeta\) and \(\delta\) are three nonnegative constant parameters with \(\epsilon > 0\). With these settings and after a few computations, we directly approach the following conclusion.

**Proposition III.1** The equilibrium, denoted by \(E(v_0, w_0)\), of model (S25) becomes:

\[
(v_0, w_0) = \left(-\zeta, \zeta^3/3 - \zeta + I\right), \quad (v_0, w_0) = \left((3I - 3\zeta)^2, (3I - 3\zeta)^2/3 + \zeta\right)
\]

when, respectively, \(\delta = 0\) and \(\delta = 1\). Generally, when \(\delta \neq 0\), model (S25) has a unique equilibrium if and only if \(D \equiv 9(I - \zeta/\delta)^2 - 4(1 - 1/\delta^3) > 0\).

Once the equilibrium is obtained, it is natural to have the following results on the oscillations generated by the model (S25) through a route of Hopf bifurcation \([9]\).

**Proposition III.2** Assume that \(\delta^3 \epsilon < 1\) and \(\delta \epsilon < 1\), and set \(s = (1 - \delta \epsilon)^{1/2}\). Then model (S25) undergoes a Hopf bifurcation at the equilibrium \(E\) when \(I\) is around either \(I_1 \equiv (s + \zeta)/\delta + s^3/3 - s\) or \(I_2 \equiv (-s + \zeta)/\delta - s^3/3 + s\). Correspondingly, the first Lyapunov coefficients for both \(I_{1,2}\) are the same, i.e.

\[
l_1 = \frac{\delta^2(-\delta \epsilon - 1)^2 + 1 - \epsilon}{(1 - \delta^2 \epsilon)^2}.
\]

Consequently, \(l_1 < 0\) as \(\delta^2 \epsilon > 2\delta - 1\), which implies that a stable oscillation is generated around \(I_{1,2}\) through a route of supercritical Hopf bifurcation.

Actually, according to the bifurcation theory \([9]\) and through a tedious computation, we around the equilibrium \(E\) obtain the FHN’s normal form which reads

\[
\dot{z} = (a + id)z + c_1|z|^2z + O(|z|^4),
\]

(S26)

where \(z = x + iy\),

\[
a = \frac{-v_0^2 - \delta \epsilon + 1}{2}, \quad d = \frac{\sqrt{-(v_0^2 - \delta \epsilon - 1)^2 + 4\epsilon}}{2},
\]

\[
\text{Re}[c_1] = -\frac{\rho^2}{8d^4} + \frac{v_0^2 \rho^2(-a \rho^2 + ad^2 + 4d^2 \rho)}{16(a^2 + d^2)d^4} + \frac{av_0^2 \rho^2(a^2 + d^2)}{(a^2 + 9d^2)d^4},
\]

and

\[
\text{Im}[c_1] = \frac{\rho^3}{8d^3} + \frac{2v_0^2 \rho^2(d^2 - \rho^2 - a \rho)}{(a^2 + d^2)d^3} - \frac{av_0^2 \rho^2(a^2 + d^2)}{(a^2 + 9d^2)d^3}.
\]
Particularly, \( \text{Re}(c_1) = l_1 \) when \( I = I_1 \) or \( I = I_2 \), as concluded in Proposition III.2 above. Here, \( \rho = (-\nu_0^2 + \delta \epsilon + 1)/2 \) and the frequency of the oscillation can be represented by

\[
f \doteq \frac{1}{2\pi} \left[ d - a \frac{\text{Im}[c_1]}{\text{Re}[c_1]} \right].
\]

Moreover, since we only focus on the local asymptotical stability of the controlled FHNM, it is sufficient for us to consider a linearization of the normal form (S26). In fact, the linearization of the normal form can be obtained directly by using the linear transformation \((x, y)^T = T(v - \nu_0, w - w_0)^T\) to model (S25), where the matrix

\[
T = \begin{pmatrix}
\frac{1}{2\rho} & -\rho \\
0 & 0
\end{pmatrix}.
\]

Hence, we obtain the controlled FHNM in a matrix form:

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} =
\begin{pmatrix}
a & -d \\
d & a
\end{pmatrix}
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} +
\begin{pmatrix}
m & -n \\
n & m
\end{pmatrix}
\begin{pmatrix}
x(t - \tau) \\
y(t - \tau)
\end{pmatrix}.
\]

Clearly, this equation is a particular case of system (S1). Moreover, \( f, a \) and \( d \) obtained above can be regarded as functions of the input stimulus current \( I \). Figures 12(a) & 12(b) shows these functions when the parameters are set as \( \zeta = 0.02, \delta = 0.004, \epsilon = 200 \). In particular, \( a = a(I) \) is slightly above zero for \( I \in (90, I_1) \) with \( I_1 = 116.386 \) and then model (S25) can produce stable oscillations. As shown in Fig. 12(a), despite of the different values, \( f \) and \( d \) have the same variation tendency as \( I \in (90, I_1) \). This implies that in such an interval of the input current, \( d \) almost determines the frequency variation of the produced stable oscillations. Moreover, in model (S27), the control gain matrix in front of the feedback control term with time delays has the following characters: (1) \( n \neq 0 \) corresponds to an asymmetric control, but \( n = 0 \) to a symmetric (diagonal) control. (2) \( m < 0 \) corresponds to a stable control; conversely, \( m > 0 \) invites an unstable control.

![FIG. S-12. The plots of the functions \( f = f(I) \), \( a = a(I) \) and \( d = d(I) \) as the input stimulus current \( I \in (0, 500) \) (a) and \( I \in (90, I_1) \) (b) with \( I_1 = 116.386 \). (c) The FHNM (S25) cannot be stabilized by a symmetric control with time delay for any \( m \). The colors in the \( I-m \) plane represent the positive exponential rates of the divergence of the trajectories generated by the controlled model (S28). Here, \( \tau = 0.1 \) and \( I \in (90, 120) \).](image)

Using the inverse transformation of \( T \) to model (S27) yields the controlled FHNM in a form of the original variables as follows:

\[
\begin{pmatrix}
\dot{\psi} \\
\dot{\omega}
\end{pmatrix} =
\begin{pmatrix}
\nu - \frac{\nu_0}{\tau} - w + I \\
\epsilon \nu + \zeta - \delta \omega
\end{pmatrix} +
\begin{pmatrix}
m + \frac{\nu_0}{\tau} - \frac{1}{\rho} n \\
\frac{d^2 \psi}{d\tau^2} - m - \frac{\nu_0}{\tau} n
\end{pmatrix}
\begin{pmatrix}
\nu(t - \tau) - \nu_0 \\
w(t - \tau) - w_0
\end{pmatrix}.
\]

Clearly, the control gain matrix here still preserves the corresponding characters (1) and (2) as illustrated above for the gain matrix of model (S27).

Finally, according to the NS condition of Case I listed in Tab. I, we can conclude that, for any \( m \), the symmetric control is useless to suppress the oscillation when \( \tau = 0.1, \zeta = 0.02, \delta = 0.004, \epsilon = 200 \), and \( I \in (90, I_1) \). This is because \( d(I) \in \mathcal{F}_I \doteq \left( \frac{\pi - \nu_0}{\tau}, \frac{\pi - \nu_0}{\zeta} \right) \) for \( I \in (90, I_1) \) and for the parameters and time delay set above. The corresponding numerical validation is shown in Figs. 12(c) & 13(b). However, if we choose a suitable asymmetric control and time delay, it is possible to achieve a successful OS for the FHNM. This is verified by numerical simulations in Figs. 13(a) & 13(c).
FIG. S-13. (a) The FHN M (S25) can be stabilized by the asymmetric control with time delay. The colors in the m-n plane represent the exponential rates of the divergence or convergence of the trajectories generated by the controlled FHN M (S28). Here, $\tau = 0.1$ and $I = 110$. (b) An unsuccessful OS in the controlled FHN M (S28) as a symmetric and stable gain matrix ($m = -4$) is used. (c) A successful OS as an unstable and asymmetric gain matrix ($m = 4, n = -5$) is used.

B. Oscillation death and synchronization in complex networks

We consider a complex system of $N$ identical nonlinear dynamical units connected through time delayed couplings:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} x_j(t - \tau), \quad i = 1, \cdots, N,$$

where $x_i \in \mathbb{R}^q$ represents the state variable of each unit and $x_0 = 0$ is assumed as the equilibrium of each unit. In addition, $(c_{ij})_{N \times N} \in \mathbb{R}^{N \times N}$ is a diagonalizable coupling matrix whose row sum is supposed to be $m$ for each $i$. Here, $m$ could be either zero or non-zero. In particular, $m = 0$ corresponds to the conventional couplings which are non-invasive to the dynamics of uncoupled unit within the synchronization manifold (SM).

Oscillation death. By using the standard technique developed in [10], we conclude directly that a study of the oscillation death (OD) in system (S29) is equivalent to an investigation of the asymptotical stability of a group of $N$ variational equations (VEs):

$$\dot{\xi}_\sigma(t) = Df(0)\xi_\sigma(t) + \lambda_\sigma \xi_\sigma(t - \tau), \quad \sigma = 1, \cdots, N,$$

where each $\xi_\sigma$ is a $q$-dimensional variable along different direction. It is easy to see that $\lambda_1 = m$ is the eigenvalue of $(c_{ij})_{N \times N}$ associated to perturbations at $x_0$ within the SM and $\lambda_{2, \cdots, N}$ are the transversal eigenvalues of $(c_{ij})_{N \times N}$.

More concretely, we suppose that each uncoupled unit satisfies the normal form of the supercritical-Hopf-bifurcation, so that $q = 2$, $Df(0) = [a, \cdots, -d, d, a]$, and the coupling matrix in front of the delay term can be written as:

$$\begin{bmatrix}
    k \cos \psi & -k \sin \psi \\
    k \sin \psi & k \cos \psi
\end{bmatrix} \begin{bmatrix}
    \lambda_\sigma & 0 \\
    0 & \lambda_\sigma
\end{bmatrix},$$

where $d$ determines the oscillation frequency, $f = (d + \gamma a)/(2\pi)$, of the uncoupled unit, and the coupling matrix could be regarded as a symmetric gain matrix for Eq. (S30) due that it is in the diagonal form. To investigate the stability, we only need to analyze the characteristic equations of Eq. (S30) for $\sigma = 1, \cdots, N$.

For the first VE [i.e., Eq. (S30) as $\sigma = 1$], the characteristic equation of is the same as Eq. (S2) as $k = \lambda_1 = m$ and $\psi = 0$. Thus, according to the NS conditions established in Tab. 1 for the symmetric control ($\psi = 0$) with time delay, despite $A = \alpha A < 1$, for any $m$ the first VE is unstable and so OD cannot be observed physically when the oscillation frequency $f$ of the uncoupled unit is in the intervals

$$\hat{J}_f = \left( \frac{p \pi - \alpha_1 + \gamma A}{2\pi \tau}, \frac{p \pi - \alpha_0 + \gamma A}{2\pi \tau} \right).$$

However, the OD can be observed only when the frequency of the uncoupled unit satisfies $f \notin \hat{J}_f$. Moreover, the stability of the transversal VEs [i.e., Eq. (S30) as $\sigma = 2, \cdots, N$, should be guaranteed through adjusting
the eigenvalues $\lambda_\sigma$ of $\{c_{ij}\}$ in light of the NS conditions. For some particular frequencies, the transversal eigenvalues might satisfy $\text{Re}(\lambda_\sigma) > 0$, which invites unstable gain matrix.

**Oscillation synchronization.** We also can discuss periodic orbit synchronization in system (S29) by using the NS condition established above. For a clear illustration, we suppose the uncoupled unit to be a three-dimensional system having a stable periodic orbit $S(t)$ with period $T$. The Floquet exponents of $S(t)$ thus are $\mu_1 = 0$ and $\mu_{2,3} = a \pm id$ ($a < 0$) [4]. We further set $\lambda_1 = m = 0$. It is easy to see that $S(t)$ is within the SM, and that the other transversal VEs become:

$$\dot{\xi}_\sigma(t) = \mathcal{D} f(S(t))\xi_\sigma(t) + \lambda_\sigma \xi_\sigma(t - \tau), \quad (S31)$$

which is different from Eq. (S30) since the synchronization of the periodic orbit $S(t)$ is considered here. Now, according to the Floquet theory [4], for each $\sigma$ and corresponding linear system

$$\dot{\xi}_\sigma(t) = \mathcal{D} f(S(t))\xi_\sigma(t), \quad (S32)$$

there exists a time-varying transformation $\xi_\sigma = P(t)\eta_\sigma$ such that system (S32) can be equivalently transformed into a linear system

$$\dot{\eta}_\sigma(t) = R\eta_\sigma(t).$$

Here, $P(t) = \Phi(t)e^{-Rt}$ is periodic with a period $T$, $\Phi(t)$ is a fundamental solution matrix of the linear system (S32), and $R$ is a nonsingular constant matrix, having eigenvalues $\mu_{1,2,3}$ and satisfying $\Phi(t + T) = \Phi(t)e^{RT}$. Then, $P^{-1}(t)$ exists and also is periodic with a period $T$. Now, we still use the transformation $\xi_\sigma = P(t)\eta_\sigma$ to every transversal VE (S31), and thus obtain

$$\dot{\eta}_\sigma(t) = R\eta_\sigma(t) + \lambda_\sigma P^{-1}(t)P(t - \tau)\eta_\sigma(t - \tau),$$

which further becomes:

$$\dot{\eta}_\sigma(t) = R\eta_\sigma(t) + \lambda_\sigma \eta_\sigma(t - \tau), \quad (S33)$$

if $\tau = pT$ with $p \in \mathbb{N}$, because of the periodicity of $P(t)$. Also we can find a nonsingular matrix to diagonalize the matrix $R$. Hence, the above transformed VE (S33) reads:

$$\dot{\eta}_\sigma(t) = \text{diag}(\mu_1, \mu_2, \mu_3)\eta_\sigma(t) + \lambda_\sigma \eta_\sigma(t - \tau), \quad (S34)$$

where, for simplicity, the notation $\eta_\sigma$ for the state variable is preserved although an additional transformation obtained from diagonalizing the matrix $R$ is used on VE (S33).

Now, a study of the synchronization of the periodic orbit in system (S29) becomes an investigation of the asymptotical stability of Eq. (S34) for $\sigma = 2, \cdots, N$ and $\tau = pT$. For a given $\sigma$, there actually exist three characteristic equations of Eq. (S34): one is

$$\lambda - \lambda_\sigma e^{-\tau} = 0,$$

and the other two are

$$\lambda - \mu_j - \lambda_\sigma e^{-\tau} = 0, \quad j = 2, 3.$$

We thus can study the root distribution of the above three characteristic equations for every $\sigma$, and derive some conditions for stability and synchronization according to the NS conditions for Case II ($a = 0$) and Case III ($a < 0$) established in Tab. I. For example, if all $\lambda_{2,\cdots,N}$ take real values, $-d\tau$ (mod $\pi$) $\in [0, \frac{\pi}{2})$, and $\tau = pT$ for $p \in \mathbb{N}$, one possible condition for synchronization of periodic orbit becomes $\lambda_{2,\cdots,N} \in (-\frac{\pi}{2\tau}, 0) \cap (-\frac{B_{0a}}{\tau}, \frac{B_{1a}}{\tau})$.


