EINSTEIN FOUR-MANIFOLDS WITH SELF-DUAL WEAHLY CURVATURE OF NONNEGATIVE DETERMINANT

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ABSTRACT. We prove that simply connected Einstein four-manifolds of positive scalar curvature are conformally Kähler if and only if the determinant of the self-dual Weyl curvature is positive.

1. Introduction

This is a sequel to the author’s thesis [16] (see also [18]) and [17, 19, 20]. The question that when a four-manifold with a complex structure admits a compatible Einstein metric of positive scalar curvature has been answered by Tian [15] (see also Odaka, Spotti, and Sun [12]), LeBrun [7], respectively. Kähler-Einstein four-manifolds of positive scalar curvature [15, 12] are \( \mathbb{C}P^2 \), \( \mathbb{C}P^1 \times \mathbb{C}P^1 \), or \( \mathbb{C}P^2 \# k \mathbb{C}P^2 \) \((3 \leq k \leq 8)\). Hermitian, Einstein four-manifolds of positive scalar curvature [7] are either Kähler-Einstein, or \( \mathbb{C}P^2 \# \mathbb{C}P^2 \) with Page metric [13], or \( \mathbb{C}P^2 \# 2 \mathbb{C}P^2 \) with Chen-LeBrun-Weber metric [2]. Recall that a Hermitian, Einstein metric is an Einstein metric which is Hermitian with respect to some integrable complex structure.

It is natural to ask, conversely,

**Question.** When does a four-manifold with an Einstein metric of positive scalar curvature admit a compatible complex structure?

There have been several answers to this question. A classical result of Derdziński (Theorem 2 in [3]) states that, passing to a double cover of the manifold if necessary, if the self-dual Weyl curvature \( W^+ \) is parallel and \( \# \text{spec}(W^+) = 2 \), then the metric is Kähler; if \( \# \text{spec}(W^+) = 2 \), then the metric is Hermitian, where \( \# \text{spec}(W^+) \) is the number of distinct eigenvalues of \( W^+ \).

Richard and Seshadri [14], Fine, Krasnov, and Panov [5], and the author [19] proved that if the metric has half nonnegative isotropic curvature, then it is either half conformally flat or Kähler. LeBrun [8] proved that if \( W^+(\omega, \omega) > 0 \) for some \( \omega \in H^2_+(M) \), then the metric is Hermitian. The author [19] proved that if the metric has conformally half nonnegative isotropic curvature, then it is either half conformally flat or Hermitian.

The eigenvalues of \( W^+ \) of any Kähler metric on four-manifolds are \(-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\), where \( R \) is the scalar curvature. LeBrun [9] proved that any Hermitian, Einstein metric of positive scalar curvature on four-manifolds must be conformal to an extremal Kähler metric, so the eigenvalues of \( W^+ \) are \(-\lambda, -\lambda, 2\lambda \) for some positive function \( \lambda \), hence \( \det W^+ > 0 \). In this paper we prove

\[ \text{det} W^+ > 0. \]

**Date:** September 2, 2019.

**2010 Mathematics Subject Classification.** Primary 53C25, 53C24, 53C55.

**Key words and phrases.** Einstein four-manifold, Weitzenböck formula, self-dual Weyl curvature, conformally Kähler metric, subharmonic function, refined Kato inequality.
**Theorem 1.1.** Simply connected Einstein four-manifolds of positive scalar curvature are conformally Kähler if and only if $\det W^+ > 0$.

On Riemannian four-manifolds, $W^+$ is traceless, so $W^+$ satisfies a simple algebraic inequality $3\sqrt{6} |W^+| \leq |W^+|^3$, and the equality holds if and only if $\#\text{spec}(W^+) \leq 2$. The idea of proving Theorem 1.1 is to prove that if $\det W^+ > 0$ then $3\sqrt{6} \det W^+ \equiv |W^+|^3$, then apply the aforementioned results of Derdziński [3] and LeBrun [9].

**Remark 1.1.** According to Theorem 2 in [3], the “simply connected” condition in Theorem 1.1 can be replaced by “oriented and $H_1(M, \mathbb{Z}_2) = 0$”.

The idea of the proof is motivated by previous work of Gursky and LeBrun [6], Yang [21], and the author [17] on the rigidity of Einstein four-manifolds of positive sectional curvature, in which the authors analyzed $|W^+|^2$, and reduced the problem to $W^+ \equiv 0$, then applied a classical result of Hitchin (Theorem 13.30 in [1]). As the author observed in Section 5 of [17], these methods might be in some sense constrained by the refined Kato inequality of Gursky and LeBrun [6]. The new idea in this paper is to analyze both $|W^+|^2$ and $\det W^+$, and, instead of reducing to $W^+ \equiv 0$, we reduce the problem to $3\sqrt{6} \det W^+ \equiv |W^+|^3$, as explained above.

The key step in the proof is to construct a subharmonic function of the form $f(|W^+|^2, \det W^+)$, which is based on Derdziński’s derivation [3] of the Weitzenböck formula for the self-dual Weyl curvature, and the author’s work [17] on an alternative proof of the refined Kato inequality, and the classification of Einstein four-manifolds of three-nonnegative curvature operator. Precisely we have

**Theorem 1.2.** Let $(M, g)$ be a compact oriented four-manifold with $\delta W^+ = 0$. If $\det W^+ > 0$, then there exists a constant $k_0$ depending on $\min_M |W^+|-3 \det W^+$, $\min_M |W^+|-2 \det W^+$, and $\min_M R$, such that for any $k \geq k_0$,

$$F_k = |W^+|^{\frac{1}{2}} \left[ 1 - 54 \left( \frac{\det W^+}{|W^+|^3} \right)^2 \right]^k$$

is a subharmonic function on $M$. Furthermore by the Stokes Theorem we get that $3\sqrt{6} \det W^+ \equiv |W^+|^3$.

Interestingly, $F_k$ is closely related to the refined Kato inequality, see Remark 2.2 in Section 2 for details.

By similar arguments we have,

**Theorem 1.3.** Simply connected Einstein four-manifolds of positive scalar curvature and $\det W^+ \geq 0$ are either anti-self-dual or conformally Kähler.

**Theorem 1.4.** Compact oriented Ricci-flat four-manifolds with $H^1(M, \mathbb{Z}_2) = 0$ and $\det W^+ \geq 0$ are anti-self-dual, therefore the universal cover of $M$ is either $\mathbb{R}^4$ with flat metric or a $K3$ surface with Calabi-Yau metric.

Theorem 1.1 and its proof suggest us to ask the following question,

**Question.** Are simply connected Einstein four-manifolds of positive scalar curvature conformally Kähler, if the self-dual Weyl curvature is nonvanishing?

We would like to point out that recently LeBrun [10] gave an alternative proof of Theorem 1.1 based on the method in [8]. Furthermore he relaxed the condition in Theorem 1.1 to $W^+ \neq 0$ and $|W^+|-3 \det W^+ \geq -\frac{5\sqrt{2}}{21\sqrt{21}}$. 

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Remark 1.2. We observe that on Einstein four-manifolds of positive scalar curvature, either $W^\pm \equiv 0$ or the average of $\det W^\pm$ has a positive lower bound. Recall the Weitzenböck formula of Derdziński [1, 3],

$$\Delta |W^\pm|^2 = 2|\nabla W^\pm|^2 + R|W^\pm|^2 - 36 \det W^\pm.$$

In our paper, we use $\Delta f = \text{tr} \nabla^2 f = g^{ij} \nabla_i \nabla_j f$ for $f \in C^\infty(M)$. Gursky and LeBrun [6] proved that either $W^\pm \equiv 0$ or $\int_M |W^\pm|^2 dv \geq \int_M \frac{R^3}{24} dv$. Combining the two formulas together we get, either $W^\pm \equiv 0$, or

$$\int_M \det W^\pm dv \geq 2 \int_M \frac{R^3}{128} dv.$$

Acknowledgement. The author thanks his advisors Professors Xianzhe Dai and Guofang Wei for their guidance, encouragement, and constant support. The author thanks Professors Claude LeBrun and Yuan Yuan for helpful discussions. The author thanks the anonymous referee for many suggestions that greatly improve the presentation of the paper. The author acknowledges the hospitality of Capital Normal University, East China Normal University, and Mathematical Sciences Research Institute, where part of this work was carried out. The author was partially supported by NSFC No.11701093 and China recruit program for global young talents.

2. Proof

We explain the method of constructing subharmonic functions of the form $f(|W^+|^2, \det W^+)$ on $M$ in two steps.

Step 1. We briefly recall Derdziński’s derivation of the Weitzenböck formula for Riemannian metrics of $\delta W^\pm = 0$ on four-manifolds.

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of $W^+$, with corresponding orthogonal eigenvectors

$$\omega_1 = e^1 \wedge e^2 + e^3 \wedge e^4, \quad \omega_2 = e^1 \wedge e^3 + e^4 \wedge e^2, \quad \omega_3 = e^1 \wedge e^4 + e^2 \wedge e^3,$$

then $W^+$ can be expressed as

$$W^+ = \frac{1}{2}(\lambda_1 \omega_1 \otimes \omega_1 + \lambda_2 \omega_2 \otimes \omega_2 + \lambda_3 \omega_3 \otimes \omega_3).$$

Let $M_W$ be the open dense subset of $M$, consisting of points at which the number of distinct eigenvalues of $W^+$ is locally constant, then $\lambda_i$ and $\omega_i$ ($i = 1, 2, 3$) may be assumed differentiable in a neighborhood of any point $p \in M_W$, so there exist 1-forms $a, b, c$ defined near $p$, such that

$$\nabla \omega_1 = a \otimes \omega_2 - c \otimes \omega_3,$$

$$\nabla \omega_2 = b \otimes \omega_3 - a \otimes \omega_1,$$

$$\nabla \omega_3 = c \otimes \omega_1 - b \otimes \omega_2.$$
By analyzing the Ricci identities for \(\omega_1, \omega_2, \omega_3\), Derdziński proved that, if \(\delta W^+ = 0\), then in a neighborhood of \(p \in M_W\),
\[
\nabla \lambda_1 = (\lambda_2 - \lambda_1)(t_a \# \omega_3)\# + (\lambda_3 - \lambda_1)(t_c \# \omega_2)\#, \\
\nabla \lambda_2 = (\lambda_1 - \lambda_2)(t_a \# \omega_3)\# + (\lambda_3 - \lambda_2)(t_b \# \omega_1)\#, \\
\nabla \lambda_3 = (\lambda_1 - \lambda_3)(t_c \# \omega_2)\# + (\lambda_2 - \lambda_3)(t_b \# \omega_1)\#, \\
\Delta \lambda_1 = 2(\lambda_1 - \lambda_2)(t_a \# \omega_3)\#^2 + 2(\lambda_1 - \lambda_3)(t_c \# \omega_2)\#^2 + \frac{R}{2} \lambda_1 - 2\lambda_1^2 - 4\lambda_2 \lambda_3, \\
\Delta \lambda_2 = 2(\lambda_2 - \lambda_1)(t_a \# \omega_3)\#^2 + 2(\lambda_2 - \lambda_3)(t_b \# \omega_1)\#^2 + \frac{R}{2} \lambda_2 - 2\lambda_2^2 - 4\lambda_1 \lambda_3, \\
\Delta \lambda_3 = 2(\lambda_3 - \lambda_1)(t_c \# \omega_2)\#^2 + 2(\lambda_3 - \lambda_2)(t_b \# \omega_1)\#^2 + \frac{R}{2} \lambda_3 - 2\lambda_3^2 - 4\lambda_1 \lambda_2,
\]
where \(\iota\) is the interior product, \('#'\) is the sharp operator.

**Remark 2.1.** Derdziński also derived the formula for \(\nabla W^+\), combining these formulas together he proved the classical Weitzenböck formula,
\[
\Delta |W^+|^2 = 2|\nabla W^+|^2 + R|W^+|^2 - 36 \det W^+.
\]

Step 2. We reduce the subharmonicity of functions of the form \(f(|W^+|^2, \det W^+)\) on \(M\) to a system of partial differential inequalities on \(\mathbb{R}^2\), based on the author's alternative proof of the refined Kato inequality.

There are only two nontrivial elementary symmetric polynomials of \(\lambda_1, \lambda_2, \lambda_3:\)
\[
\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 = 0, \\
x \triangleq -2\sigma_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = |W^+|^2, \\
y \triangleq \sigma_3 = \lambda_1 \lambda_2 \lambda_3 = \det W^+.
\]
For simplicity, we define vector fields \(X \triangleq (\lambda_1 - \lambda_2)(t_a \# \omega_3)\#, Y \triangleq (\lambda_2 - \lambda_3)(t_b \# \omega_1)\#, \)
\(Z \triangleq (\lambda_3 - \lambda_1)(t_c \# \omega_2)\#\) in a neighborhood of \(p \in M_W\). We have
\[
\nabla x = -2(\lambda_1 - \lambda_2)X - 2(\lambda_2 - \lambda_3)Y - 2(\lambda_3 - \lambda_1)Z, \\
\nabla y = \lambda_3(\lambda_1 - \lambda_2)X + \lambda_1(\lambda_2 - \lambda_3)Y + \lambda_2(\lambda_3 - \lambda_1)Z, \\
\Delta x = 8|X|^2 + 8|Y|^2 + 8|Z|^2 - 4\langle X, Y \rangle - 4\langle X, Z \rangle - 4\langle Y, Z \rangle + (Rx - 36y), \\
\Delta y = -4\lambda_3|X|^2 - 4\lambda_1|Y|^2 - 4\lambda_2|Z|^2 - 4\lambda_2\langle X, Y \rangle - 4\lambda_1\langle X, Z \rangle - 4\lambda_3\langle Y, Z \rangle \\
+ \left(\frac{3}{2} Ry - x^2\right).
\]
Let \(f = f(x, y)\) be a differentiable function on \(M\). On \(M_W\), we have
\[
\Delta f = f_x \Delta x + f_y \Delta y + f_x x |\nabla x|^2 + f_y y |\nabla y|^2 + 2 f_{xy} \nabla x \nabla y \\
= [8f_x - 4\lambda_3 f_y + (\lambda_1 - \lambda_2)^2(4f_{xx} + \lambda_1^2 f_{yy} - 4\lambda_3 f_{xy})]|X|^2 \\
+ [8f_x - 4\lambda_1 f_y + (\lambda_2 - \lambda_3)^2(4f_{xx} + \lambda_2^2 f_{yy} - 4\lambda_1 f_{xy})]|Y|^2 \\
+ [8f_x - 4\lambda_2 f_y + (\lambda_3 - \lambda_1)^2(4f_{xx} + \lambda_3^2 f_{yy} - 4\lambda_2 f_{xy})]|Z|^2 \\
+ 2[-2f_x - 2\lambda_2 f_y + (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(4f_{xx} + \lambda_1 \lambda_3 f_{yy} + 2\lambda_2 f_{xy})]|\langle X, Y \rangle| \\
+ 2[-2f_x - 2\lambda_1 f_y + (\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(4f_{xx} + \lambda_2 \lambda_3 f_{yy} + 2\lambda_1 f_{xy})]|\langle X, Z \rangle| \\
+ 2[-2f_x - 2\lambda_3 f_y + (\lambda_3 - \lambda_1)(\lambda_1 - \lambda_2)(4f_{xx} + \lambda_1 \lambda_2 f_{yy} + 2\lambda_3 f_{xy})]|\langle Y, Z \rangle| \\
+ (Rx - 36y)f_x + \left(\frac{3}{2} Ry - x^2\right)f_y.
\]
We denote $A$, $B$, $C$ as the coefficients of $|X|^2$, $|Y|^2$, $|Z|^2$ in Equation (1), respectively; and $2D$, $2E$, $2F$ as the coefficients of $\langle X, Y \rangle$, $\langle X, Z \rangle$, $\langle Y, Z \rangle$ in Equation (1), respectively. We define

$$I \triangleq A|X|^2 + B|Y|^2 + C|Z|^2 + 2D\langle X, Y \rangle + 2E\langle X, Z \rangle + 2F\langle Y, Z \rangle,$$

$$II \triangleq (Rx - 36y)f_x + \left(\frac{3}{2}Ry - x^2\right)f_y.$$

Then we have

$$\Delta f = I + II.$$

If $I \geq 0$ and $II \geq 0$ on $M_W$, then $\Delta f \geq 0$ on $M_W$, moreover since $M_W$ is an open dense subset of $M$ and $f$ is differentiable, we conclude that $\Delta f \geq 0$ on $M$.

We consider $I$ as a quadratic form of (components of) $X, Y, Z$. In order for $I \geq 0$, we need $A > 0$, $B > 0$, $C > 0$. Consider $I$ as a quadratic function of (components of) $X$, then its minimum is

$$\tilde{I} = A^{-1}[(AB - D^2)|Y|^2 + (AC - E^2)|Z|^2 + 2(AF - DE)\langle Y, Z \rangle].$$

In order for $\tilde{I} \geq 0$, we need $AB - D^2 > 0$, $AC - E^2 > 0$. Consider $\tilde{I}$ as a quadratic function of (components of) $Y$, then its minimum is

$$(AB - D^2)^{-1}(ABC - AF^2 - BE^2 - CD^2 + 2DEF)|Z|^2.$$

Therefore the quadratic form $I \geq 0$ if $A, B, C, D, E, F$ satisfy the following system:

$$\begin{cases} A > 0, & B > 0, & C > 0, \\ I_{31} \triangleq AB - D^2 > 0, \\ I_{32} \triangleq AC - E^2 > 0, \\ I_{33} \triangleq BC - F^2 > 0, \\ I_4 \triangleq ABC - AF^2 - BE^2 - CD^2 + 2DEF \geq 0. \end{cases}$$

Notice that the characterization of the quadratic form $I > 0$ follows from Sylvester’s criterion.

Observe that $A + B + C > 0$, $I_{31} + I_{32} + I_{33} > 0$, and $I_4 \geq 0$ will ensure that all $A, B, C, I_{31}, I_{32}, I_{33}$, are positive. Therefore $f$ is a subharmonic function on $M$ if

$$\begin{cases} 0 \leq I_1 \triangleq II = (Rx - 36y)f_x + \left(\frac{3}{2}Ry - x^2\right)f_y; \\ 0 < I_2 \triangleq A + B + C; \\ 0 < I_3 \triangleq AB + AC + BC - D^2 - E^2 - F^2; \\ 0 \leq I_4 = ABC - AF^2 - BE^2 - CD^2 + 2DEF. \end{cases}$$

Plugging in $A, B, C, D, E, F$ to the above system, we conclude that $f(x, y)$ is a subharmonic function on $M$, if $f(x, y)$, considering as a function on $\Omega = \{(x, y) \in \mathbb{R}^2 :$
\[ x^3 \geq 54y^2 \} \subset \mathbb{R}^2, \] satisfies the following system of partial differential inequalities,

\[
\begin{aligned}
0 \leq I_1 &= (Rx - 36y)f_x + \left( \frac{2}{3} Ry - x^2 \right)f_y, \\
0 < I_2 &= 24f_{xx} + 72f_{xy} + x^2f_{yy} + 48f_x, \\
0 < I_3 &= 6(x^3 - 54y^2)(f_{xx}f_{yy} - f_{xy}^2) + 24(7f_{xx} - 12f_y)f_{xx} + 8(63f_{xx} - 4x^2f_y)f_{xy} + x(7f_{xx} - 12f_y)f_{yy} + 180f_x^2 - 12x^2f_y, \\
0 \leq I_4 &= 6(x^3 - 54y^2)(f_{xx}f_{yy} - f_{xy}^2)f_x + 4(30x^3 - 72yf_{xx}f_{yy} - x^2f_y)f_{xx} + 4(90yf_{xx}^2 - 4x^2f_xf_y - 3xyf_y^2)f_{xy} + (5x^2f_x^2 - 12xyf_{xx}f_y - 9y^2f_y^2)f_{yy} + 100f_x^3 - 14xf_y^2 - 8yf_y^3.
\end{aligned}
\]

**Proof** of Theorem 1.2. We will construct a function \( f(x, y) \) that satisfies System (PDI) in the subregion \( \Omega_\delta = \{ y \geq \delta > 0, x^3 \geq 54y^2 \} \subset \Omega. \)

Define \( z = x^2 \frac{1}{3} y \in [-\frac{1}{3\sqrt{6}}, \frac{1}{3\sqrt{6}}] \) at points where \( x \neq 0, \) and \( f(x, y) = x^3(1 - 54z^2)^k, \) plugging \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \) into System (PDI), we have

\[
\begin{aligned}
I_1 &= \frac{1}{6} x^3(1 - 54z^2)^k [R + 36(18k - 1)x^{-1}y], \\
I_2 &= \frac{2}{3} x^2(1 - 54z^2)^{-1} [54(18k - 1)(18k + 7)z^2 - (162k - 7)], \\
I_3 &= \frac{2}{9} x^{-\frac{5}{3}}(1 - 54z^2)^{2k-2} [2916(18k - 1)(18k + 5)z^4 - 108(1944k^2 - 162k + 5)z^2 \\
&\quad - (162k - 5)], \\
I_4 &= 0.
\end{aligned}
\]

Since \( M \) is compact and \( x^3 \geq 54y^2, \) if \( y \geq \delta \) for some \( \delta > 0, \) then \( z \geq \delta_1, \) \( x^{-1}y \geq \delta_2, \) for some \( \delta_1 > 0, \) \( \delta_2 > 0. \) By choosing \( k \) large enough, we get that \( I_i \geq 0, \) moreover \( I_i > 0 \) when \( 1 - 54z^2 > 0, \) \( i = 1, 2, 3. \) So we have \( I \geq 0 \) and \( II \geq 0 \) on \( M_W, \) therefore \( \Delta f \geq 0 \) on \( M. \)

By Stokes Theorem we get \( \Delta f \equiv 0 \) on \( M, \) then \( I \equiv 0, \) \( II \equiv 0 \) on \( M_W. \) From \( II \equiv 0 \) on \( M_W \) we get that \( 1 - 54z^2 \equiv 0 \) on \( M_W, \) which implies \( 3\sqrt{6}y \equiv x^3 \) on \( M, \)

**Proof** of Theorem 1.1. By Theorem 1.2 and the aforementioned results in [3, 9], \((M, g)\) is conformally Kähler.

**Remark 2.2.** Recall the refined Kato inequality [6] for \( W^+ \) of Einstein metrics on four-manifolds,

\[
|\nabla W^+|^2 \geq \frac{5}{3} |\nabla|W^+||^2.
\]

Consider a function \( f(x) = f(|W^+|^2), \) by the Weitzenböck formula, we have

\[
\Delta f = f' \Delta x + f'' |\nabla x|^2 \\
= 2|\nabla W^+|^2 f' + 4|\nabla|W^+||^2 f'' + (Rx - 36y)f'.
\]

Denote \( \Delta_D f = 2|\nabla W^+|^2 f' + 4x|\nabla|W^+||^2 f'' \), the “derivative part” of the Weitzenböck formula, then the refined Kato inequality for \( W^+ \) can be interpreted as

\[
\Delta_D x^\frac{1}{6} \geq 0.
\]

Moreover, \( \frac{1}{6} \) is the smallest power such that this inequality holds, see Section 5 in [17] for details. The function we construct, \( x^\frac{1}{6}(1 - 54z^2)^k, \) can be considered as
a homogeneous variation of \(x^\lambda\), since \(z\) depends only on the quotient \(\lambda \over \lambda_2\), but is independent of the magnitude of \(W^2\).

**Proof** of Theorem 1.3. We will construct a function \(f(x, y)\) that satisfies System (PDI) in the subregion \(\Omega_0 = \{x > 0, y \geq 0, x^3 \geq 54y^2\} \subset \Omega\).

Consider \(f = x^3h(z), z \in [0, \frac{1}{3\sqrt{6}}]\), with \(h(z) \geq 0\) to be determined. Plugging \(f_x, f_y, f_{xx}, f_{xy}, f_{yy}\) into System (PDI), we have

\[
I_1 = \frac{1}{6} x^{-1} \left[ (Rx - 36y)h - 6x^2 (1 - 54z^2)h' \right].
\]

\[
I_2 = \frac{1}{3} x^{-\frac{5}{6}} \left[ 3(1 - 54z^2)h'' - 270zh' + 14h \right].
\]

\[
I_3 = \frac{1}{9} x^{-\frac{5}{2}} \left[ -27z(1 - 54z^2)h' h'' + 3(1 - 54z^2)hh'' - 6(2 - 351z^2)h^2 - 324zh'h' + 10h^2 \right].
\]

\[I_4 \equiv 0.\]

Suppose \(h'(z) = (1 - 54z^2)^{-1} \phi(z)h(z)\) with \(\phi(z)\) to be determined, then we have

\[h'' = (1 - 54z^2)^{-2} \left[ (1 - 54z^2)\phi' + \phi^2 + 108z\phi \right] \phi \cdot h.\]

Plugging into the above system, we have

\[
I_1 = \frac{1}{6} x^{\lambda} \left[ R - 6x^2 (\phi + 6z) \right].
\]

\[
I_2 = \frac{1}{3} x^{\lambda - \frac{3}{2}} \left[ 3(1 - 54z^2)\phi' + 3\phi^2 + 54z\phi + 14(1 - 54z^2) \right].
\]

\[
I_3 = \frac{1}{9} x^{\lambda - 2} \left[ (1 - 54z^2)^{-2} \left[ 3(1 - 54z^2)(1 - 54z^2)\phi' - 27z\phi^3 - 9(1 + 108z^2)\phi^2 + 10(1 - 54z^2)^2 \right] \right].
\]

First notice that \(\max_{\Omega_0} x^{\lambda} = \infty\), so \(I_1 \geq 0\) in \(\Omega_0\) if and only if \(\phi + 6z \leq 0\). It is obvious that if \(\phi + 6z \leq 0\), then \(1 - 54z^2 - 9z\phi \geq 0\) when \(z \geq 0\).

Next notice that \((1 - 54z^2 - 9z\phi)hI_2 - I_3 = 12\phi^2 h^2 + 4(1 - 54z^2 - 9z\phi)^2 h^2\), so if \(I_3 \geq 0\) in \(\Omega_0\) and \(\phi + 6z \leq 0\) then \(I_2 \geq 0\) in \(\Omega_0\).

In summary, if \(I_1 \geq 0\) and \(\phi + 6z \leq 0\), then \(I_1 \geq 0\) and \(I_2 \geq 0\). \(I_3 \geq 0\) and \(\phi + 6z \leq 0\) is equivalent to an Abel differential inequality of the second kind [11] on \([0, \frac{1}{3\sqrt{6}}]\) with a constraint condition,

\[
3(1 - 54z^2)(1 - 54z^2 - 9z\phi)\phi' - 27z\phi^3 - 9(1 + 108z^2)\phi^2 + 10(1 - 54z^2)^2 \geq 0,
\]

\[
\phi + 6z \leq 0.
\]

To further simplify the system, we denote \(\psi(z) = \phi(z) + 6z\), then the constrained Abel differential inequality (2) can be written as

\[
3(1 - 54z^2)(1 - 9z\psi)\psi' - 27z\psi^3 - 9(1 + 54z^2)\psi^2 + 270z\psi - 8(1 + 54z^2) \geq 0,
\]

\[
\psi \leq 0.
\]
We choose the initial value $\psi(0) = -6\sqrt{6}$, then the constrained Abel differential inequality (3) has a solution $\psi(z)$ on $[0, \frac{1}{3\sqrt{6}}]$, which is monotonically increasing and $\psi(\frac{1}{3\sqrt{6}}) = -3\sqrt{6}$. By the definition of $\psi(z)$, one can check that $h(z)$ is monotonically decreasing on $[0, \frac{1}{3\sqrt{6}}]$ and $h(\frac{1}{3\sqrt{6}}) = 0$. So we have $I_i \geq 0$, moreover $I_i > 0$ when $1 - 54z^2 > 0$, $i = 1, 2, 3$. Therefore $I_1 \geq 0$, $I_2 \geq 0$ on $M_W \{W^+ = 0\}$, and $\Delta f \geq 0$ on $M_W \{W^+ = 0\}$.

Furthermore, by the definition of $\phi$, the function $f$ we construct satisfies the property that $f^{-k} = x^k h^{-k}(z) \in C^2(M)$ for sufficient large $k$. By the above argument, we have $\Delta f^{(k)} \geq 0$ on $M$. By Stokes Theorem, we get $\Delta f^{(k)} \equiv 0$, then $I \equiv 0$, $\Pi \equiv 0$ on $M_W$. From $\Pi \equiv 0$ on $M_W$, we get that either $x = 0$ or $1 - 54z^2 = 0$ on $M_W$, therefore, either $x = 0$, or $x^3 = 54y^2$ and $y > 0$ on $M$. By Prop 5 in [3], we get that either $x \equiv 0$, or $x^3 \equiv 54y^2$ and $y > 0$ on $M$. Therefore $(M, g)$ is either anti-self-dual or conformally Kähler.

\[ \square \]

Proof of Theorem 1.4. Notice that the nonnegativity of $I$ is independent of the sign of the scalar curvature. By the same argument as in the proof of Theorem 1.3, if $R = 0$, then $I_i = -hx^2(\phi + 6z) = -hx^2\psi$. Therefore $\psi$ satisfies the same Abel differential inequality with the same constraint condition, and we get the same conclusion that either $x \equiv 0$, or $x^3 \equiv 54y^2$ and $y > 0$ on $M$.

If $x^3 \equiv 54y^2$ and $y > 0$, then $(M, g)$ is conformally Kähler. By Proposition 5 in [3], $\bar{g} = (24x)^{\frac{4}{3}} g$ is a Kähler metric with scalar curvature $\bar{R} = (24x)^{\frac{2}{3}} > 0$. On the other hand, by the conformal change of the scalar curvature, $\bar{R}$ has to be nonpositive somewhere, which leads to a contradiction.

Therefore we have $x \equiv 0$, that is, $(M, g)$ is anti-self-dual.

\[ \square \]

Remark 2.3. It is interesting to observe in the proof of Theorem 1.3 that all homogeneous variations of $x^k$, that is, functions of the form $x^k h(z)$, where $h(z)$ is an arbitrary differentiable function, solve the partial differential equation

\[
0 = I_4 = 6(x^3 - 54y^2)(f_{xx}f_{yy} - f_{xy}^2)f_x + 4(30x^2f_x^2 - 72y f_x f_y - x^2 f_y^2)f_{xx}
\]

\[
+ 4(90y^2 f_x^2 - 4x^2 f_y f_{xx} - 3xy f_x f_{xy}^2)f_{xy} + (5x^2 f_x^2 - 12xy f_x f_y - 9y^2 f_y^2)f_{yy}
\]

\[
+ 100f_x^3 - 14x f_x f_y^2 - 8y f_y^3.
\]

One may ask whether this equation admits solutions of a different form, which may help us to characterize Kähler-Einstein or Hermitian, Einstein metrics using different curvature conditions by constructing functions $f(x, y)$ that satisfies System (PDI) in different subsets of $\Omega$.

References


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