A NOTE ON THE EXACT CONTROLLABILITY FOR NONAUTONOMOUS HYPERBOLIC SYSTEMS

TATSUEN LI AND ZHIQIANG WANG∗

School of Mathematical Sciences, Fudan University
Handan Road, No. 220, Shanghai, 200433, China

Abstract. By choosing suitable examples, we show that, quite different from the autonomous hyperbolic case, the exact boundary controllability for nonautonomous hyperbolic systems possesses various possibilities.

1. Introduction. There are numerous results on the exact boundary controllability for autonomous hyperbolic systems (In the linear case, see [1-2]; while, in the semi-linear or quasi-linear case, see [3-8, 10-11]). These results show that for autonomous hyperbolic systems, if there is no zero eigenvalue, for any given initial data at \( t = 0 \) and any final data at \( t = T \), one can always choose suitable boundary controls such that the classical solution to the corresponding mixed initial-boundary value problem with the initial data at \( t = 0 \) takes exactly the final data at \( t = T \), provided that \( T > 0 \) is suitable large. By translation, this conclusion is still valid for any given initial data at \( t = t_0 \) and any final data at \( t = t_0 + T \). Thus, in the autonomous case, the controllability time \( T \) can be taken as a constant for all \( t_0 \in \mathbb{R} \).

For nonautonomous hyperbolic systems, the exact boundary controllability should depend on the choice of the initial time \( t_0 \). This leads to the following questions:

1. Is it possible to have the exact boundary controllability only for some initial time \( t_0 \)? In particular, is it possible to lose the exact boundary controllability for all \( t_0 \in \mathbb{R} \) or have the exact boundary controllability for all \( t_0 \in \mathbb{R} \)?

2. Does the controllability time depend on the initial time \( t_0 \) in general? Namely, is there the exact boundary controllability only for \( T > T(t_0) \), where \( T(t_0) \) is a function of \( t_0 \)?

More particularly, is it possible that the controllability time is independent of \( t_0 \)?

In this paper, by choosing suitable examples to answer the previous questions, we will find that, quite different from the autonomous hyperbolic case, the exact boundary controllability for nonautonomous hyperbolic systems possesses various possibilities, then it should be studied in a more delicate way (cf. [9]).

2. Case of two-side control. Consider the following first order nonautonomous linear hyperbolic system:

\[
\begin{align*}
\frac{\partial r}{\partial t} - f'(t) \frac{\partial r}{\partial x} &= 0, \\
\frac{\partial s}{\partial t} + f'(t) \frac{\partial s}{\partial x} &= 0,
\end{align*}
\]

2000 Mathematics Subject Classification. Primary: 93B05, 35L50, 49J20.

Key words and phrases. Nonautonomous hyperbolic system, mixed initial-boundary value problem, exact boundary controllability.

∗ Corresponding author.

Supported by the Special Funds for Major State Basic Research Projects of China and the Specialized Research Fund for the Doctoral Program of Higher Education.
where $t$ and $x$ are independent variables, $r$ and $s$ are unknown functions of $(t, x)$, $f'(t)$ is the derivative of a given smooth function $f(t)$ with
\[
f'(t) > 0, \quad \forall t \in \mathbb{R}.
\] (2.2)

Obviously, (2.1) is a first order linear strictly hyperbolic system with coefficients depending on $t$.

Without loss of generality, we take the following boundary conditions:
\[
x = 0: \quad r + s = g(t),
\]
\[
x = L: \quad r - s = h(t),
\]
where $g$ and $h$ are $C^1$ functions of $t$. In the case of two-side control, both $g$ and $h$ will be taken as boundary controls(cf. [6, 8-9]); while, in the case of one-side control, $g$ or $h$ will be taken as boundary control(cf. [7-9]).

We give the initial condition
\[
t = t_0: \quad (r, s) = (r_0(x), s_0(x)) \quad (0 \leq x \leq L)
\] (2.5)
and the final condition
\[
t = t_0 + T: \quad (r, s) = (r_T(x), s_T(x)) \quad (0 \leq x \leq L),
\] (2.6)
where $(r_0(x), s_0(x))$ and $(r_T(x), s_T(x))$ are all $C^1$ vector functions on $0 \leq x \leq L$.

In this section we first consider the case of two-side control. Noting that by means of the invertible transformation
\[
\bar{t} = f(t),
\] (2.7)
system (2.1) can be reduced to the following autonomous hyperbolic system with constant coefficients
\[
\begin{cases}
\frac{\partial r}{\partial \bar{t}} - \frac{\partial r}{\partial x} = 0, \\
\frac{\partial s}{\partial \bar{t}} + \frac{\partial s}{\partial x} = 0.
\end{cases}
\] (2.8)

By [6] and [8], there is the two-side exact boundary controllability on the interval $[t_0, t_0 + T]$ if and only if the maximum determinate domain for the forward Cauchy problem of system (2.1) with the initial data (2.5) does not intersect the maximum determinate domain for the backward Cauchy problem of system (2.1) with the final data (2.6).

The forward characteristic passing through the point $(t_0, 0)$, given by
\[
\begin{cases}
\frac{dx}{dt} = f'(t), \\
t = t_0: \quad x = 0,
\end{cases}
\] (2.9)
has the equation
\[
x = f(t) - f(t_0).
\] (2.10)
While, the backward characteristic passing through the point $(t_0, L)$, given by
\[
\begin{cases}
\frac{dx}{dt} = -f'(t), \\
t = t_0: \quad x = L,
\end{cases}
\] (2.11)
has the equation
\[ x = L + f(t_0) - f(t). \]  
(2.12)

(2.10) and (2.12) are the left and right boundaries of the maximum determinate domain for the forward Cauchy problem of system (2.1) with the initial data at \( t = t_0 \), respectively. The \( t \)-coordinate \( \xi \) of the intersection point of these two characteristics should satisfy
\[ f(\xi) = f(t_0) + \frac{L}{2}. \]  
(2.13)

If \( f(t_0) + \frac{L}{2} \) does not belong to the range of \( f \), these two characteristics can not intersect each other in a finite time, then there is no exact boundary controllability; otherwise, we get
\[ \xi = f^{-1}\left(f(t_0) + \frac{L}{2}\right). \]  
(2.14)

Similarly, the backward characteristic passing through the point \((t_0 + T, 0)\), given by
\[ \left\{ \begin{array}{l} \frac{dx}{dt} = -f'(t), \\ t = t_0 + T: x = 0, \end{array} \right. \]  
(2.15)

has the equation
\[ x = f(t_0 + T) - f(t). \]  
(2.16)

While, the forward characteristic passing through the point \((t_0 + T, L)\), given by
\[ \left\{ \begin{array}{l} \frac{dx}{dt} = f'(t), \\ t = t_0 + T: x = L, \end{array} \right. \]  
(2.17)

has the equation
\[ x = L + f(t) - f(t_0 + T). \]  
(2.18)

(2.16) and (2.18) are the left and right boundaries of the maximum determinate domain for the backward Cauchy problem of system (2.1) with the final data at \( t = t_0 + T \), respectively. The \( t \)-coordinate \( \tau \) of the intersection point of these two characteristics should satisfy
\[ f(\tau) = f(t_0 + T) - \frac{L}{2}. \]  
(2.19)

If \( f(t_0 + T) - \frac{L}{2} \) does not belong to the range of \( f \), these two characteristics can not intersect each other in a finite time, then there is no exact boundary controllability; otherwise, we get
\[ \tau = f^{-1}\left(f(t_0 + T) - \frac{L}{2}\right). \]  
(2.20)

Thus, two maximum determinate domains do not intersect each other if and only if both \( f(t_0) + \frac{L}{2} \) and \( f(t_0 + T) - \frac{L}{2} \) belong to the range of \( f \) and
\[ \tau > \xi, \]  
(2.21)

namely,
\[ f(t_0 + T) - f(t_0) > L. \]  
(2.22)
Noting (2.2), it is easy to see that if (2.22) holds, then \( f(t_0) + \frac{L}{2} \) and \( f(t_0 + T) - \frac{L}{2} \) belong to the range of \( f \). Hence, we have

**Proposition 2.1.** For the nonautonomous hyperbolic system (2.1), under assumption (2.2), there is the two-side exact boundary controllability on the interval \([t_0, t_0 + T]\) if and only if (2.22) holds.

When the range of \( f \) is \( \mathbb{R} \) or \((a, +\infty)\), where \( a \) is a real number, (2.22) can be equivalently rewritten as

\[ T > f^{-1}(f(t_0) + L) - t_0 , \tag{2.23} \]

Thus, we have

**Corollary 2.1.** Under the assumptions of Proposition 2.1, if the range of \( f \) is \( \mathbb{R} \) or \((a, +\infty)\), where \( a \) is a real number, then for any given \( t_0 \in \mathbb{R} \), we have the two-side exact boundary controllability on the interval \([t_0, t_0 + T]\), where

\[ T > T(t_0) \equiv f^{-1}(f(t_0) + L) - t_0 . \tag{2.24} \]

(2.24) tells us that the controllability time depends on \( t_0 \) in general. However, if we assume that \( f'(t) \) is a \( T_0 \)-periodic function and

\[ \int_0^{T_0} f'(t) \, dt = L , \tag{2.25} \]

then (2.22) holds for all \( t_0 \in \mathbb{R} \), provided that \( T > T_0 \). Hence, we have

**Corollary 2.2.** Under the assumptions of Corollary 2.1, suppose furthermore that \( f'(t) \) is a \( T_0 \)-periodic function and (2.25) holds. Then for any given \( t_0 \in \mathbb{R} \), we have the two-side exact boundary controllability on the interval \([t_0, t_0 + T]\) with \( T > T_0 \). In this special case, the controllability time can be taken to be independent of \( t_0 \).

When the range of \( f \) is \((-\infty, b)\), where \( b \) is a real number, in order to have (2.22), \( t_0 \) should satisfy

\[ f(t_0) < b - L , \quad i.e., \quad t_0 < f^{-1}(b - L) , \tag{2.26} \]

Hence, we have

**Corollary 2.3.** Under the assumptions of Proposition 2.1, if the range of \( f \) is \((-\infty, b)\), where \( b \) is a real number, then only for such \( t_0 \) that (2.26) holds, one can get the two-side exact boundary controllability on the interval \([t_0, t_0 + T]\), where \( T \) satisfies (2.24). Thus, for nonautonomous hyperbolic systems, it is possible to have the exact boundary controllability only for a part of initial time \( t_0 \).

When the range of \( f \) is \((a, b)\), where \( a \) and \( b \) are two real numbers with \( a < b \), it is easy to get that when

\[ b - a > L , \tag{2.27} \]

only for such \( t_0 \) that (2.26) holds, there exists \( T > 0 \), given by (2.24), such that we have (2.22); while when

\[ b - a \leq L , \tag{2.28} \]
one can never get (2.22) for any \( t_0 \in \mathbb{R} \) and \( T > 0 \). Hence, we have

**Corollary 2.4.** Under the assumptions of Proposition 2.1, if the range of \( f \) is \((a, b)\), where \( a \) and \( b \) are two real numbers with \( a < b \), then when (2.27) holds, only for such \( t_0 \) that (2.26) holds, one can get the two-side exact boundary controllability on the interval \([t_0, t_0 + T]\), where \( T \) satisfies (2.24); while, when (2.28) holds, there is no two-side exact boundary controllability for any \( t_0 \in \mathbb{R} \).

### 3. Case of one-side control.

We now consider the one-side exact boundary controllability. Without loss of generality, we assume that the control is acted only on \( x = L \). By [7-8], there is the one-side exact boundary controllability on the interval \([t_0, t_0 + T]\) if and only if the maximum determinate domain for the forward one-side mixed initial-boundary value problem of system (2.1) with the initial data (2.5) and the boundary condition (2.3) at \( x = 0 \) does not intersect the maximum determinate domain for the backward one-side mixed initial-boundary value problem of system (2.1) with the final data (2.6) and the boundary condition (2.3) at \( x = 0 \).

The backward characteristic (2.12) passing through the point \((t_0, L)\) and the \( t \)-axis are the right and left boundaries, respectively, of the maximum determinate domain for the forward one-side mixed problem. The \( t \)-coordinate \( \xi \) of the intersection point of these two boundaries should satisfy

\[
f(\xi) = f(t_0) + L. \tag{3.1}
\]

When \( f(t_0) + L \) does not belong to the range of \( f \), the characteristic (2.12) never intersects the \( t \)-axis, then there is no one-side exact boundary controllability; otherwise, we get

\[
\xi = f^{-1}(f(t_0) + L). \tag{3.2}
\]

Similarly, the forward characteristic (2.18) passing through \((t_0 + T, L)\) and the \( t \)-axis are the right and left boundaries, respectively, of the maximum determinate domain for the backward one-side mixed problem. The \( t \)-coordinate \( \tau \) of the intersection point of these two boundaries should satisfy

\[
f(\tau) = f(t_0 + T) - L. \tag{3.3}
\]

When \( f(t_0 + T) - L \) does not belong to the range of \( f \), the characteristic (2.18) never intersects the \( t \)-axis, then there is no one-side exact boundary controllability; otherwise, we get

\[
\tau = f^{-1}(f(t_0 + T) - L). \tag{3.4}
\]

In order to ensure that these two maximum determinate domains never intersects each other, we should ask

\[
\tau > \xi, \tag{3.5}
\]

namely,

\[
f(t_0 + T) - f(t_0) > 2L. \tag{3.6}
\]

Noting (2.2), when (3.6) holds, both \( f(t_0) + L \) and \( f(t_0 + T) - L \) belong to the range of \( f \). Similarly to Proposition 2.1 and Corollaries 2.1-2.4, we have

**Proposition 3.1.** For the nonautonomous hyperbolic system (2.1), under assumption (2.2), there is the one-side exact boundary controllability on the interval \([t_0, t_0 + T]\) if and only if (3.6) holds.
Corollary 3.1. Under the assumptions of Proposition 3.1, if the range of \( f \) is \( \mathbb{R} \) or \((a, +\infty)\), where \( a \) is a real number, then for any given \( t_0 \in \mathbb{R} \), we have the one-side exact boundary controllability on the interval \([t_0, t_0 + T]\), where
\[
T > T(t_0) \triangleq f^{-1}(f(t_0) + 2L) - t_0.
\] (3.7)

Corollary 3.2. Under the assumptions of Corollary 3.1, suppose furthermore that \( f'(t) \) is a \( T_0 \)-periodic function and
\[
\int_0^{T_0} f'(t) \, dt = 2L,
\] (3.8)
holds. Then for any given \( t_0 \in \mathbb{R} \), we have the one-side exact boundary controllability on the interval \([t_0, t_0 + T]\) with \( T > T_0 \).

Corollary 3.3. Under the assumptions of Proposition 3.1, if the range of \( f \) is \((-\infty, b)\), where \( b \) is a real number, then only for such \( t_0 \) that
\[
f(t_0) < b - 2L, \quad \text{i.e.,} \quad t_0 < f^{-1}(b - 2L)
\] (3.9)
holds, one can get the one-side exact boundary controllability on the interval \([t_0, t_0 + T]\), where \( T \) satisfies (3.7).

Corollary 3.4. Under the assumptions of Proposition 3.1, if the range of \( f \) is \((a, b)\), where \( a, b \) are two real numbers with \( a < b \), then when
\[
b - a > 2L,
\] (3.10)
only for such \( t_0 \) that (3.9) holds, one can get the one-side exact boundary controllability on the interval \([t_0, t_0 + T]\), where \( T \) satisfies (3.7); while, when
\[
b - a \leq 2L,
\] (3.11)
there is no one-side exact boundary controllability for any \( t_0 \in \mathbb{R} \).

4. Remark Similar results hold(cf. [10-11]) for the one-dimensional nonautonomous linear wave equation
\[
u_{tt} - (f'(t))^2 u_{xx} = 0.
\] (4.1)

REFERENCES

CONTROLLABILITY FOR NONAUTONOMOUS HYPERBOLIC SYSTEMS


Received January 2006; revised August 2006.

E-mail address: dqli@fudan.edu.cn; wzq@fudan.edu.cn