A NOTE ON THE ONE-SIDE EXACT BOUNDARY OBSERVABILITY FOR QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract. The known theory on the one-side exact boundary observability for first order quasilinear hyperbolic systems requires that the unknown variables are suitably coupled or satisfy the Group Property in boundary conditions on the non-observation side (see [1]–[2], [11]). In this paper we illustrate, with an inspiring example, that the one-side exact boundary observability can be realized by means of a suitable coupling of the unknown variables in quasilinear hyperbolic system itself instead of in boundary conditions. Moreover, an implicit duality between the one-side exact boundary controllability and one-side exact boundary observability is also revealed in this situation.

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1. Introduction

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \quad (1.1)$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of $(t, x)$; $A(u)$ is an $n \times n$ matrix with smooth entries $a_{ij}(u)$ ($i, j = 1, \ldots, n$); $F(u) = (f_1(u), \ldots, f_n(u))^T$ is a smooth vector function of $u$ and

$$F(0) = 0. \quad (1.2)$$

By hyperbolicity, for any given $u$ on the domain under consideration, $A(u)$ has $n$ real eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$ and a complete set of left eigenvectors $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ ($i = 1, \ldots, n$):

$$l_i(u)A(u) = \lambda_i(u)l_i(u). \quad (1.3)$$

Suppose that there are no zero eigenvalues:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \ldots, m; \; s = m + 1, \ldots, n). \quad (1.4)$$

For simplicity, we assume without loss of generality that the number of positive eigenvalues is equal to that of negative ones:

$$n - m = m, \; \text{i.e.,} \; n = 2m. \quad (1.5)$$
Let 

$$v_i = l_i(u)u \quad (i = 1, \ldots, n).$$

(1.6)

$v_i$ is called the diagonal variable corresponding to the $i$-th eigenvalue $\lambda_i(u)$.

The boundary conditions are given as follows:

$$x = 0 : \quad v_s = G_s(t, v_1, \ldots, v_m) + H_s(t) \quad (s = m + 1, \ldots, n),$$

(1.7)

$$x = L : \quad v_r = G_r(t, v_{m+1}, \ldots, v_n) + H_r(t) \quad (r = 1, \ldots, m),$$

(1.8)

where $L$ is the length of the interval $0 \leq x \leq L$, $G_i$ ($i = 1, \ldots, n$) are smooth functions, and without loss of generality we assume that

$$G_i(t, 0, \ldots, 0) \equiv 0 \quad (i = 1, \ldots, n).$$

(1.9)

We arbitrarily give the initial condition

$$t = 0 : \quad u = \varphi(x), \quad 0 \leq x \leq L,$$

(1.10)

where $\|\varphi\|_{C^1[0,L]}$ is sufficiently small and the conditions of $C^4$ compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$, respectively.

The possibility of realizing the one-side exact boundary observability for (1.1) by boundary observations only at $x = 0$ was discussed in [1]–[2]. The result obtained there can be described as follows:

Suppose that in a neighbourhood of $u = 0$, the boundary condition (1.8) on the non-observation side $x = L$ can be equivalently rewritten as

$$x = L : \quad v_s = \overline{G}_s(t, v_1, \ldots, v_m) + \overline{H}_s(t) \quad (s = m + 1, \ldots, n)$$

(1.11)

with

$$\overline{G}_s(t, 0, \ldots, 0) \equiv 0 \quad (s = m + 1, \ldots, n).$$

(1.12)

If

$$T > L \left( \max_{r=1,\ldots,m} \frac{1}{|\lambda_r(0)|} + \max_{s=m+1,\ldots,n} \frac{1}{\lambda_s(0)} \right),$$

(1.13)

and $H_i$ ($i = 1, \ldots, n$) are all given with small $C^1[0,T]$ norm, then we can uniquely determine the initial data $\varphi$ by means of the observed boundary values of the diagonal variables $v_r = \overline{v}_r(t)$ ($r = 1, \ldots, m$) corresponding to the negative eigenvalues at $x = 0$ on the time interval $[0, T]$. Moreover, the following observability estimate holds:

$$\|\varphi\|_{C^1[0,L]} \leq C \left( \sum_{r=1}^{m} \|\overline{v}_r\|_{C^1[0,T]} + \sum_{i=1}^{n} \|H_i\|_{C^1[0,T]} \right),$$

(1.14)

here and hereafter $C$ denotes a positive constant.

The condition (1.11) guarantees the well-posedness of the corresponding backward mixed initial-boundary value problem and is a generalization of the Group Condition in the linear case (see [11]). This condition indicates that in order to realize the one-side exact boundary observability by boundary observations only at $x = 0$, the diagonal variables $v_s$ ($s = m + 1, \ldots, n$) corresponding to the positive eigenvalues and the diagonal variables $v_r$ ($r = 1, \ldots, m$) corresponding to the negative eigenvalues should be coupled in a suitable way in the boundary condition (1.8) on the non-observation side $x = L$. 
Similarly, if \( v_s \) (\( s = m + 1, \ldots, n \)) and \( v_r \) (\( r = 1, \ldots, m \)) are suitably coupled in the boundary condition (1.7) on \( x = 0 \) so that in a neighbourhood of \( u = 0 \) (1.7) can be equivalently rewritten as

\[
    x = 0 : \quad v_r = G_r(t, v_{m+1}, \ldots, v_n) + H_r(t) \quad (r = 1, \ldots, m)
\]

with

\[
    G_r(t, 0, \ldots, 0) \equiv 0 \quad (r = 1, \ldots, m),
\]

then we can obtain the one-side exact boundary observability by boundary observations at \( x = L \).

If neither (1.7) on \( x = 0 \) nor (1.8) on \( x = L \) has such kind of coupling property, we cannot obtain the one-side exact boundary observability directly from [1]–[2] even in the linear case. Nevertheless, in this paper we illustrate with an inspiring example that the one-side exact boundary observability can be realized once the unknown variables are suitably coupled in the quasilinear hyperbolic system (1.1) itself.

To show this, we consider the following quasilinear hyperbolic system

\[
\begin{align*}
    r_t + \lambda(r,s) r_x &= f(r,s), \\
    s_t + \mu(r,s) s_x &= g(r,s),
\end{align*}
\]

where \( \lambda, \mu, f \) and \( g \) are suitably smooth functions with

\[
    \lambda(0,0) < 0 < \mu(0,0)
\]

and

\[
    f(0,0) = g(0,0) = 0.
\]

The given initial condition is

\[
    t = 0 : \quad (r,s) = (r_0(x), s_0(x)), \quad 0 \leq x \leq L,
\]

while the boundary conditions are given as

\[
\begin{align*}
    x = 0 : \quad s &= H(t), \\
    x = L : \quad r &= \overline{H}(t),
\end{align*}
\]

where the unknown variables \( r \) and \( s \) are decoupled.

However, the one-side local exact boundary observability can be realized once \( r \) and \( s \) are suitably coupled in the system (1.17) and \( T > 0 \) is large enough. Here, the suitability of coupling means that the right-hand term \( f(r,s) \) of the first equation in (1.17) essentially depends on \( s \):

\[
    f_s(0,0) \neq 0;
\]

or the right-hand term \( g(r,s) \) of the second equation in (1.17) essentially depends on \( r \):

\[
    g_r(0,0) \neq 0.
\]
Theorem 1 (One-side observation at \( x = L \)). Suppose that \( \lambda, f \in C^1, \mu, g \in C^2 \) and (1.18)–(1.19) hold. Suppose furthermore that (1.24) holds and
\[
T > L \left( \frac{1}{|\lambda(0,0)|} + \frac{1}{\mu(0,0)} \right). \tag{1.25}
\]
For any given initial data \( (r_0, s_0) \) and boundary functions \( (H, \overline{H}) \), if \( \|(r_0, s_0)\|_{C^1[0,L] \times C^2[0,L]} \) and \( \|(H, \overline{H})\|_{C^2[0,T] \times C^1[0,T]} \) are sufficiently small, and the conditions of \( C^2 \) compatibility and of \( C^1 \) compatibility are satisfied at the points \((t, x) = (0,0)\) and \((0, L)\) respectively, then the initial data \((r_0, s_0)\) can be uniquely determined by means of the observed boundary value \( s = \overline{s}(t) \) at \( x = L \) and the given boundary functions \((H, \overline{H})\) on the interval \([0, T]\). Moreover, the following observability estimate holds:
\[
\|(r_0, s_0)\|_{C^1[0,L] \times C^2[0,L]} \leq C \left( \|\overline{s}\|_{C^2[0,T]} + \|(H, \overline{H})\|_{C^2[0,T] \times C^1[0,T]} \right). \tag{1.26}
\]

Remark 1. The estimate (1.25) on the exact observability time \( T \) in Theorem 1 coincides with (1.13). Moreover, this estimate is also sharp.

Remark 2. Suppose that \( \lambda, f \in C^2, \mu, g \in C^1 \) and (1.23) holds instead of (1.24), then the one-side local exact boundary observability can be similarly realized by observing the value \( r = \overline{r}(t) \) at \( x = 0 \).

In order to prove Theorem 1, with the help of the coupling condition (1.24), the one-side exact boundary observability problem for the original first order quasilinear hyperbolic system (1.17) is reduced to the corresponding observability problem for a second order quasilinear hyperbolic equation in Section 2. In Section 3, we establish the corresponding one-side local exact boundary observability for the latter problem. Finally, in Section 4, comparing the corresponding result on the one-side exact boundary controllability under some similar hypotheses (see [7]), we show that there is an implicit duality between the exact controllability and the exact observability in this situation.

2. REDUCTION OF THE PROBLEM

By (1.24) and noting (1.19), it is easy to see that the second equation in (1.17) can be equivalently rewritten as
\[
r = R(s, s_x, s_t) \tag{2.1}
\]
in a neighbourhood of \((r, s, s_x) = (0, 0, 0)\), where \( R \) is a \( C^2 \) function with
\[
R(0, 0, 0) = 0. \tag{2.2}
\]
Moreover, we have
\[
R_1 = \frac{\mu s_x - g_s}{g_r - \mu r s_x}, \quad R_2 = \frac{\mu}{g_r - \mu r s_x}, \quad R_3 = \frac{1}{g_r - \mu r s_x}, \tag{2.3}
\]
where \( R_1, R_2 \) and \( R_3 \) stand for the derivatives of \( R(s, s_x, s_t) \) with respect to the variables \( s, s_x \) and \( s_t \), respectively.
Substituting (2.1) into the first equation in (1.17) and noting (2.3), we obtain the following second order partial differential equation for $s$:

$$s_{tt} + (\lambda + \mu)s_{tx} + \lambda \mu s_{xx} = F,$$

(2.4)

where

$$F = F(s, s_x, s_t) \overset{\text{def.}}{=} (g_s - \mu s_{sx})(s_t + \lambda s_x) + f(g_r - \mu r_s)$$

(2.5)

and $r = R(s, s_x, s_t)$. Obviously, $F$ is a $C^1$ function with

$$F(0, 0, 0) = 0.$$  

(2.6)

Noting (1.18), the discriminant

$$\Delta = (\lambda + \mu)^2 - 4\lambda \mu = (\mu - \lambda)^2 > 0$$

(2.7)

in a neighbourhood $(s, s_x, s_t) = (0, 0, 0)$, and then (2.4) is a second order quasi-linear hyperbolic equation with the following characteristics

$$\frac{dx}{dt} = \lambda (< 0) \quad \text{and} \quad \frac{dx}{dt} = \mu (> 0).$$

(2.8)

Combining (1.20) and the second equation in (1.17), we get the initial condition for $s$

$$t = 0 : \ s = \varphi(x), \ s_t = \psi(x), \quad 0 \leq x \leq L,$$

(2.9)

where

$$\begin{cases}
\varphi(x) = s_0(x), \\
\psi(x) = g(r_0(x), s_0(x)) - \mu(r_0(x), s_0(x))s'_0(x).
\end{cases}$$

(2.10)

Obviously, $(\varphi, \psi) \in C^2[0, L] \times C^1[0, L]$. Moreover, by the equivalence of (2.1) and the second equation in (1.17), we have

$$\begin{cases}
r_0(x) = R(\varphi(x), \varphi'(x), \psi(x)), \\
s_0(x) = \varphi(x),
\end{cases}$$

(2.11)

then

$$C_1 \| (r_0, s_0) \|_{C^1[0,L] \times C^2[0,L]} \leq \| (\varphi, \psi) \|_{C^2[0,L] \times C^1[0,L]} \leq C_2 \| (r_0, s_0) \|_{C^1[0,L] \times C^2[0,L]},$$

(2.12)

where $C_1$ and $C_2$ are positive constants.

In addition, by (1.21)–(1.22) and noting (2.1), the boundary conditions for $s$ are

$$x = 0 : \ s = H(t),$$

(2.13)

$$x = L : \ R(s, s_x, s_t) = \overline{H}(t).$$

(2.14)

Obviously, the conditions of $C^2$ compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$ respectively.

Under the assumptions of Theorem 1, we want to prove that the corresponding mixed initial-boundary value problem (2.4), (2.9) and (2.13)–(2.14) admits a unique semi-global $C^2$ solution $s = s(t, x)$ on the domain $R(T) = \{(t, x)|0 \leq t \leq T, 0 \leq x \leq L\}$ with small $C^2$ norm, the initial data $(\varphi, \psi)$ can be uniquely determined by means of the boundary observed value $s = \overline{s}(t)$ at $x = L$ together
with the given boundary functions \((H, \overline{H})\) on the interval \([0, T]\), moreover, the following observability inequality holds:

\[
\|(\varphi, \psi)\|_{C^2([0,L]) \times C^1([0,L])} \leq C\|\overline{s}\|_{C^2([0,T])} + \|(H, \overline{H})\|_{C^2([0,T] \times C^1([0,T])}. \tag{2.15}
\]

After that, in view of (2.1), we set

\[
r = r(t, x) \overset{\text{def}}{=} R(s(t, x), s_x(t, x), s_t(t, x)). \tag{2.16}
\]

It is easy to see that \((r, s) = (r(t, x), s(t, x))\) is a classical solution to the original mixed initial-boundary value problem (1.17) and (1.20)–(1.22) on the domain \(R(T)\) with small \(C^1 \times C^2\) norm. Then we can apply the boundary observed value \(s = \overline{s}(t)\) at \(x = L\) together with the given boundary functions \((H, \overline{H})\) on the interval \([0, T]\) to determine the unique initial data \((r_0, s_0)\), and the desired observability inequality (1.26) follows from (2.12) and (2.15). In this way, the one-side exact boundary observability will be realized for the original problem.

3. Proof of Theorem 1

Let

\[
u = u, \quad v = s_x, \quad w = s_t. \tag{3.1}
\]

Taking

\[
U = (u, v, w)^T \tag{3.2}
\]

as the new unknown variables, the mixed initial-boundary value problem (2.4), (2.9) and (2.13)–(2.14) can be reduced to the corresponding mixed problem for a first order quasilinear system.

Noting (2.3) and (1.18), we have

\[
(R_2 - \lambda R_3)(0, 0, 0) \neq 0. \tag{3.3}
\]

Then, applying directly the known theory on the semi-global \(C^1\) solution to the mixed initial-boundary value problem for first order quasilinear hyperbolic systems (see [4], [12]–[13]), we reach the conclusion as follows (cf. [7]–[8]).

**Lemma 1** (Semi-global \(C^2\) solution). For any given and possibly quite large \(T_0 > 0\), the forward mixed initial-boundary value problem (2.4), (2.9) and (2.13)–(2.14) admits a unique semi-global \(C^2\) solution \(s = s(t, x)\) on the domain \(R(T_0) = \{(t, x)|0 \leq t \leq T_0, 0 \leq x \leq L\}\) with small \(C^2\) norm, provided that \(\|(\varphi, \psi)\|_{C^2([0,L] \times C^1([0,L])}\) and \(\|(H, \overline{H})\|_{C^2([0,T_0] \times C^1([0,T_0]))}\) are small enough (depending on \(T_0\)). Moreover, the following estimate holds:

\[
\|s\|_{C^2([R(T_0)])} \leq C\|\overline{s}\|_{C^2([0,L]) \times C^1([0,L])} + \|(H, \overline{H})\|_{C^2([0,T_0] \times C^1([0,T_0]))}, \tag{3.4}
\]

where \(C\) is a positive constant (possibly depending on \(T_0\)).

**Remark 3.** In Lemma 1, if the boundary condition on \(x = L\) is taken as

\[
x = L : s = \overline{H}(t), \tag{3.5}
\]

where \(\overline{H} \in C^2\), and the conditions of \(C^2\) compatibility are satisfied at the point \((t, x) = (0, L)\). Then we can also establish the existence and uniqueness of
semi-global $C^2$ solution for the forward mixed initial-boundary value problem with the following estimate:

$$\|s\|_{C^2([0,T])} \leq C(\|\varphi\|_{C^2[0,L] \times C^1[0,L]} + \|H, \overline{H}\|_{C^2[0,T] \times C^2[0,T]}).$$

(3.6)

A similar conclusion can be reached for the backward mixed problem.

**Corollary 1.** The Cauchy problem for the second order quasilinear hyperbolic equation (2.4) with the initial condition (2.9) admits a unique global $C^2$ solution $s = s(t,x)$ on the whole maximum determinate domain and

$$\|s\|_{C^2} \leq C(\|\varphi\|_{C^2[0,L] \times C^1[0,L]}),$$

(3.7)

provided that $\|\varphi\|_{C^2[0,L] \times C^1[0,L]}$ is small enough.

**Proof of Theorem 1.** From the above, we need only to prove the corresponding one-side exact boundary observability for the mixed initial-boundary value problem for the second order quasilinear hyperbolic equation (2.4) with (2.9) and (2.13)--(2.14). That is to say: it suffices to prove that the initial data $(\varphi, \psi)$ can be uniquely determined by means of the boundary observed value $s = \overline{s}(t)$ at $x = L$ together with the given boundary functions $(H, \overline{H})$ on the interval $[0,T]$ and the observability inequality (2.15) holds.

As in [3], we will prove the desired result by means of a direct and constructive method.

For any $T > 0$ satisfying (1.25), by Lemma 1, the forward mixed initial-boundary value problem (2.4),(2.9) and (2.13)--(2.14) admits a unique $C^2$ solution $s = s(t,x)$ with small $C^2$ norm on the domain $R(T) = \{(t,x)|0 \leq t \leq T, 0 \leq x \leq L\}$ provided that $\|\varphi\|_{C^2[0,L] \times C^1[0,L]}$ and $\|H, \overline{H}\|_{C^2[0,T] \times C^1[0,T]}$ are sufficiently small. Hence the $C^2[0,T]$ norm of the boundary observed value $s = \overline{s}(t)$ defined as $s(t,L)$ at $x = L$ is also sufficiently small.

Observing that the boundary condition (2.14) comes from (1.22) and the second equation in (1.17), it is easy to see that (2.14) can be equivalently rewritten as

$$x = L: \quad s_x = \frac{g(H(t),s) - s_t}{\mu(H(t),s)}.$$  

(3.8)

Then, noting (1.18)--(1.19), for $s_x = \overline{s}_x(t) \equiv s_x(t,L)$ we have

$$\|\overline{s}_x\|_{C^1[0,T]} \leq C(\|\overline{s}\|_{C^2[0,T]} + \|\overline{H}\|_{C^1[0,T]}).$$

(3.9)

Now we interchange the status of $t$ and $x$ so that the equation (2.4) is reduced to

$$s_{xx} + \frac{1}{\lambda} s_{xt} + \frac{1}{\lambda \mu} s_t = \frac{F}{\lambda \mu}.$$  

(3.10)

Next, we solve the leftward Cauchy problem for (3.10) with the initial condition

$$x = L: \quad s = \overline{s}(t), \quad s_x = \overline{s}_x(t), \quad 0 \leq t \leq T$$  

(3.11)

(Fig. 1). By Corollary 1, this Cauchy problem admits a unique $C^2$ solution $s = s^l(t,x)$ with small $C^2$ norm on the whole maximum determinate domain $R^l$
and
\[ \|s^l\|_{C^2[R^l]} \leq C\|\overline{s}, \overline{s}_x\|_{C^2[0,T] \times C^1[0,T]} \leq C(\|\overline{s}\|_{C^2[0,T]} + \|\overline{H}\|_{C^1[0,T]}). \] (3.12)

Moreover, \( s = s'(t, x) \) is the restriction of the \( C^2 \) solution \( s = s(t, x) \) on the domain \( R(T) \cap R^l \).

Since \( T > 0 \) satisfies (1.25) and the \( C^2 \) norm of \( s'(t, x) \) is sufficiently small, the domain \( R^l \) intersects the \( t \)-axis. Hence there exists a \( T_0 \) \((0 < T_0 < T)\) such that the value of the solution \( (s, s_t) = (\Phi(x), \Psi(x)) \) \((0 \leq x \leq L)\) on \( t = T_0 \) is determined by \( s = s'(t, x) \) (Fig. 2). Consequently, we get from (3.12) that
\[ \|(\Phi, \Psi)\|_{C^2[0,L] \times C^1[0,L]} \leq C(\|\overline{s}\|_{C^2[0,T]} + \|\overline{H}\|_{C^1[0,T]}). \] (3.13)

Finally, taking
\[ t = T_0 : s = \Phi(x), s_t = \Psi(x), \quad 0 \leq x \leq L \] (3.14)
as the initial condition, (2.13) and
\[ x = L : s = \overline{s}(t) \] (3.15)
as the boundary conditions, we solve the backward mixed initial-boundary value problem for the equation (2.4) (Fig. 3). By Lemma 1 and Remark 3, \( s = s(t, x) \), as the only \( C^2 \) solution to the above backward mixed initial-boundary value problem on the domain \( R(T_0) = \{(t, x)|0 \leq t \leq T_0, 0 \leq x \leq L\} \), satisfies the estimate
\[ \|s\|_{C^2[R(T_0)]} \leq C(\|(\Phi, \Psi)\|_{C^2[0,L] \times C^1[0,L]} + \|\overline{s}\|_{C^2[0,T]} + \|H\|_{C^2[0,T]}). \] (3.16)

Then the desired observability inequality (2.15) follows immediately from (2.9) and (3.13). This finishes the proof of Theorem 1. □

4. Duality between Controllability and Observability

In the case of linear hyperbolic systems, there is a duality between controllability and observability (see [9]–[11]). The HUM (Hilbert Uniqueness Method) initiated by J.-L. Lions ([9]–[10]) essentially uses the observability inequality to obtain the controllability via the duality. While, for nonlinear hyperbolic
systems (nonlinear equations and/or nonlinear boundary conditions), there is no duality between controllability and observability. However, comparing the result on observability in this paper with the corresponding result on controllability in [7], we can easily find that in the case of the first order quasilinear hyperbolic system \((1.17)\) and \((1.20)-(1.22)\), with assumptions \((1.18)-(1.19)\) and \((1.24)\), we can realize not only the one-side exact boundary controllability by one control acting on \(x = L\), but also the one-side exact boundary observability by one observation taken at \(x = L\). Meanwhile, the requirement on the exact controllability time coincides with that on the exact observability time, and both the number of control and the number of observed value are equal to 1. This shows that, as in [1]–[2] and [5]–[6], there is still an implicit duality between controllability and observability in this exceptional case.

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