EXACT BOUNDARY CONTROLLABILITY AND
OBSERVABILITY FOR FIRST ORDER QUASILINEAR
HYPERBOLIC SYSTEMS WITH A KIND OF NONLOCAL
BOUNDARY CONDITIONS

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Abstract. In this paper we establish the theory on the semiglobal classical
solution to first order quasilinear hyperbolic systems with a kind of nonlocal
boundary conditions, and based on this, the corresponding exact boundary
controllability and observability are obtained by a constructive method. More-
over, with the linearized Saint-Venant system and the 1-D linear wave equation
as examples, we show that the number of both boundary controls and bound-
ary observations can not be reduced, and consequently, we conclude that the
exact boundary controllability for a hyperbolic system in a network with loop
can not be realized generically.

1. Introduction. Consider the following first order quasilinear hyperbolic system

\[ \frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = B(u), \]

(1.1)

where \( u = (u_1, \cdots, u_n)^T \) is the unknown vector function of \( (t,x) \), \( A(u) \) is a \( n \times n \) matrix with suitably smooth entries \( a_{ij}(u) \) \( (i,j = 1, \cdots, n) \), \( B(u) = (b_1(u), \cdots, b_n(u))^T \) is a suitably smooth vector function with

\[ B(0) = 0. \]

(1.2)

By hyperbolicity, for any given \( u \) on the domain under consideration, the matrix
\( A(u) \) possesses \( n \) real eigenvalues \( \lambda_1(u), \cdots, \lambda_n(u) \) and a complete set of left

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eigenvectors \( l_i(u) = (l_{i1}(u), \ldots, l_{in}(u)) \) \((i = 1, \ldots, n)\):

\[
l_i(u)A(u) = \lambda_i(u)l_i(u).
\]

Multiplying (1.1) with \( l_i(u) \) \((i = 1, \ldots, n)\), we obtain the characteristic form of (1.1):

\[
l_i(u) \left( \frac{\partial u}{\partial t} + \lambda_i(u) \frac{\partial u}{\partial x} \right) = f_i(u) := l_i(u)B(u) \quad (i = 1, \ldots, n).
\]

Clearly,

\[
f_i(0) = 0 \quad (i = 1, \ldots, n).
\]

In what follows, we assume that there exist \( l, m \in \mathbb{Z}, 0 \leq l \leq m \leq n \), such that on the domain under consideration

\[
\lambda_p(u) < \lambda_q(u) \equiv 0 < \lambda_{r}(u) \quad (p = 1, \ldots, l; \; q = l + 1, \ldots, m; \; r = m + 1, \ldots, n).
\]

Let us assume that the initial condition is

\[
u(0, x) = \varphi(x), \quad x \in [0, L],
\]

and the boundary conditions take the following nonlocal form:

\[
v_r(t, 0) = G_r(t, v_1(t, 0), \ldots, v_m(t, 0), v_{l+1}(t, L), \ldots, v_n(t, L)) + H_r(t) \quad (r = m + 1, \ldots, n),
\]

\[
v_p(t, L) = G_p(t, v_1(t, 0), \ldots, v_m(t, 0), v_{l+1}(t, L), \ldots, v_n(t, L)) + H_p(t) \quad (p = 1, \ldots, l),
\]

where

\[
v_i = l_i(u)u \quad (i = 1, \ldots, n),
\]

\(v_i\) being called the diagonalized variables corresponding to \( \lambda_i(u)\), \(L\) is the length of the space interval, \(G_p, G_r, H_p, H_r \) \((p = 1, \ldots, l; \; r = m + 1, \ldots, n)\) are all suitably smooth functions. Without loss of generality, we assume that

\[
G_p(t, 0, \ldots, 0) \equiv G_r(t, 0, \ldots, 0) \equiv 0 \quad (p = 1, \ldots, l; \; r = m + 1, \ldots, n).
\]

The basic features of this kind of nonlocal boundary conditions can be described as follows: on the whole boundary \((x = 0\) and \(x = L)\) of the domain under consideration, the diagonalized variables \((v_{m+1}(t, 0), \ldots, v_n(t, 0), v_1(t, L), \ldots, v_l(t, L))\) corresponding to the coming characteristics can be expressed explicitly by all the other diagonalized variables \((v_1(t, 0), \ldots, v_m(t, 0), v_{l+1}(t, L), \ldots, v_n(t, L))\). It is a generalization of the local nonlinear boundary conditions considered in [9, 14], however, the local existence and uniqueness of \(C^1\) solution to this mixed problem (1.4) and (1.7)-(1.9) can still be treated under the framework of [12]. In order to study the exponential stabilization of the \(H^2\) solution, Coron et al. [1] established the existence and uniqueness of \(H^2\) solution to this kind of mixed problem under the assumption that there are no zero eigenvalues.

In this paper, we first establish the theory on the semiglobal \(C^1\) solution to the mixed problem (1.4) and (1.7)-(1.9) in Section 2, then, under the assumption that system (1.4) possesses no zero eigenvalues, by means of a constructive method, we obtain the results on the local exact boundary controllability and observability in Section 3. Direct applications to the Saint-Venant system and the 1-D quasilinear wave equation are given in Sections 4 and 5, respectively. Finally, with the linearized Saint-Venant system and the 1-D linear wave equation as examples, we show that the number of both boundary controls and boundary observations can not be reduced,
and consequently, we conclude that the exact boundary controllability for a system in a network with loop can not be realized generically.

2. **Semiglobal \(C^1\) solution to the nonlocal mixed problem.**

**Theorem 2.1. (Semiglobal \(C^1\) solution)** Suppose that on the domain under consideration, \(l_i, \lambda_i, f_i, G_p, G_r, H_p, H_r\) \((i = 1, \cdots, n; p = 1, \cdots, l; r = m + 1, \cdots, n)\) and \(\varphi\) are all \(C^1\) functions with respect to their arguments. Suppose furthermore that (1.5)-(1.6) and (1.11) hold and the conditions of \(C^1\) compatibility are satisfied at the points \((t, x) = (0, 0)\) and \((0, L)\). For any preassigned and possibly quite large \(T > 0\), if \(\|\varphi\|_{C^1[0, L]}, \|H_p\|_{C^1[0, T]}\) \((p = 1, \cdots, l)\) and \(\|H_r\|_{C^1[0, T]}\) \((r = m + 1, \cdots, n)\) are sufficiently small (depending on \(T\)), then the exact boundary controllability for a system \((1.4)\) and \((1.7)-(1.9)\) admits a unique semiglobal \(C^1\) solution \(u = u(t, x)\) with small \(C^1\) norm on the domain \(R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}\). Moreover, when \(\frac{\partial G_p}{\partial \tau}(t, \cdot)\) \((p = 1, \cdots, l)\) and \(\frac{\partial G_r}{\partial \tau}(t, \cdot)\) \((r = m + 1, \cdots, n)\) satisfy local Lipschitz conditions with respect to the variable \(v = (v_1, \cdots, v_n)^T\), we have the following estimate

\[
\|u\|_{C^1[R(T)]} \leq C(\|\varphi\|_{C^1[0, L]} + \sum_{p=1}^{l} \|H_p\|_{C^1[0, T]} + \sum_{r=m+1}^{n} \|H_r\|_{C^1[0, T]}), \tag{2.1}
\]

where \(C\) is a positive constant possibly depending on \(T\).

**Proof.** Assume that \(u = u(t, x)\) is a \(C^1\) solution to the mixed problem \((1.4)\) and \((1.7)-(1.9)\) on \(R(T)\). Let

\[
\tilde{u}(t, x) = u(t, L - x), \quad (t, x) \in R(T). \tag{2.2}
\]

\(\tilde{u} = \tilde{u}(t, x)\) satisfies the following mixed problem on \(R(T)\):

\[
i_i(\tilde{u}) \frac{\partial \tilde{u}}{\partial \tau} - \lambda_i(\tilde{u}) \frac{\partial \tilde{u}}{\partial x} = f_i(\tilde{u}) \quad (i = 1, \cdots, n), \tag{2.3}
\]

\[
\tilde{u}(0, x) = \varphi(L - x), \quad x \in [0, L],
\]

\[
\tilde{v}_i(t, L) = G_r(t, \tilde{v}_1(t, L), \cdots, \tilde{v}_m(t, L), \tilde{v}_{t+1}(t, 0), \cdots, \tilde{v}_n(t, 0)) + H_r(t)
\]

\[
(r = m + 1, \cdots, n), \tag{2.4}
\]

\[
\tilde{v}_p(t, 0) = G_p(t, \tilde{v}_1(t, L), \cdots, \tilde{v}_m(t, L), \tilde{v}_{t+1}(t, 0), \cdots, \tilde{v}_n(t, 0)) + H_p(t)
\]

\[
(p = 1, \cdots, l), \tag{2.5}
\]

where

\[
\tilde{v}_i(t, x) = l_i(\tilde{u}(t, x))\tilde{u}(t, x) = v_i(t, L - x) \quad (i = 1, \cdots, n). \tag{2.6}
\]

Furthermore, let

\[
U = \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \in \mathbb{R}^{2n}, \tag{2.8}
\]

\[
\Lambda_i(U) = \lambda_i(u), \quad \Lambda_{n+i}(U) = -\lambda_i(\bar{u}) \quad (i = 1, \cdots, n), \tag{2.9}
\]

\[
L_i(U) = (l_i(u), 0, \cdots, 0) \in \mathbb{R}^{2n}, \quad L_{n+i}(U) = (0, \cdots, 0, l_i(\bar{u})) \in \mathbb{R}^{2n}
\]

\[
(i = 1, \cdots, n), \tag{2.10}
\]

\[
F_i(U) = f_i(u), \quad F_{n+i}(U) = f_i(\bar{u}) \quad (i = 1, \cdots, n), \tag{2.11}
\]

\[
V_j = L_j(U)U \quad (j = 1, \cdots, 2n). \tag{2.12}
\]

It is easy to see that

\[
V_i(t, x) = v_i(t, x), \quad V_{n+i}(t, x) = \bar{v}_i(t, x) = v_i(t, L - x) \quad (i = 1, \cdots, n), \tag{2.13}
\]
and \( U(t, x) = \frac{u(t, x)}{\overline{u}(t, x)} \) is the \( C^1 \) solution to the mixed problem of the following \emph{enlarged system} with \emph{local} boundary conditions on \( R(T) \):

\[
L_j(U) \left( \frac{\partial U}{\partial t} + \Lambda_j(U) \frac{\partial U}{\partial x} \right) = F_j(U) \quad (j = 1, \cdots, 2n), \tag{2.14}
\]

\[
U(0, x) = \left( \begin{array}{c} \varphi(x) \\ \varphi(L - x) \end{array} \right), \quad x \in [0, L], \tag{2.15}
\]

\[
V_r(t, 0) = G_r(t, V_1(t, 0), \cdots, V_m(t, 0), V_{n+i+1}(t, 0), \cdots, V_{2n}(t, 0)) + H_r(t)
\]

\[
(r = m + 1, \cdots, n), \tag{2.16}
\]

\[
V_{n+p}(t, 0) = G_p(t, V_1(t, 0), \cdots, V_m(t, 0), V_{n+i+1}(t, 0), \cdots, V_{2n}(t, 0)) + H_p(t),
\]

\[
(p = 1, \cdots, l), \tag{2.17}
\]

\[
V_p(t, L) = G_p(t, V_{n+m}(t, L), \cdots, V_{n+m}(t, L), V_{n+i}(t, L), \cdots, V_{n}(t, L)) + H_p(t)
\]

\[
(p = 1, \cdots, l), \tag{2.18}
\]

\[
V_{n+l}(t, L) = G_l(t, V_{n+m}(t, L), \cdots, V_{n+m}(t, L), V_{n+i}(t, L), \cdots, V_{n}(t, L)) + H_l(t)
\]

\[
(r = m + 1, \cdots, n). \tag{2.19}
\]

Since the boundary conditions in the enlarged mixed problem are all \emph{local}, the theory on the semiglobal classical solution in [9] (or [14]) can be directly applied to show that the mixed problem (2.14)-(2.19) admits a unique semiglobal \( C^1 \) solution \( U(t, x) = \left( \begin{array}{c} u(t, x) \\ \overline{u}(t, x) \end{array} \right) \) on \( R(T) \). On the other hand, noting (2.9)-(2.12), it is easy to see that \( \overline{U}(t, x) = \left( \begin{array}{c} \overline{u}(t, L - x) \\ u(t, L - x) \end{array} \right) \) is also a \( C^1 \) solution to the same mixed problem (2.14)-(2.19) on \( R(T) \). By the uniqueness of \( C^1 \) solution (cf. [12]), \( U(t, x) \equiv \overline{U}(t, x) \), then \( \overline{u}(t, x) \equiv u(t, L - x) \).

Thus, from the existence of the semiglobal \( C^1 \) solution \( U = U(t, x) \) to the \emph{enlarged} mixed problem (2.14)-(2.19) on \( R(T) \), we get immediately the existence of the semiglobal \( C^1 \) solution \( u = u(t, x) \) to the original \emph{nonlocal} mixed problem (1.4) and (1.7)-(1.9) on \( R(T) \).

Moreover, when \( \frac{\partial G_r}{\partial t}(t, \cdot) \) \((p = 1, \cdots, l)\) and \( \frac{\partial G_r}{\partial x}(t, \cdot) \) \((r = m + 1, \cdots, n)\) satisfy local Lipschitz conditions with respect to the variable \( v = (v_1, \cdots, v_n)^T \), the estimate (2.1) can be obtained directly from the above argument.

\[\square\]

**Remark 2.1.** The basic idea of the proof of Theorem 2.1 comes from the treatment in [1].

**Corollary 2.1.** Suppose that on the domain under consideration, \( l_i, \lambda_i, f_i \) \((i = 1, \cdots, n)\) and \( \varphi \) are all \( C^1 \) functions with respect to their arguments, and (1.5)-(1.6) hold. If \( \| \varphi \|_{C^1([0, L])} \) is sufficiently small, then Cauchy problem (1.4) and (1.7) admits a unique global \( C^1 \) solution \( u = u(t, x) \) on the whole maximum determinate domain \( D = \{(t, x)| t \geq 0, x_1(t) \leq x \leq x_2(t)\} \) (Fig. 1), where \( x = x_1(t) \) and \( x = x_2(t) \) are two curves defined as follows:

\[
\begin{align*}
\frac{dx_i}{dt} &= \max_{r=m+1, \cdots, n} \lambda_r(u(t, x_1)), \\
t &= 0: \quad x_1 = 0
\end{align*}
\]

\[\tag{2.20}\]
and

\[
\begin{cases}
\frac{dx_2}{dt} = \min_{p=1,\ldots,l} \lambda_p(u(t,x_2)), \\
\quad t = 0 : x_2 = L,
\end{cases}
\]  

(2.21)

respectively (see [12]). Moreover, we have the following estimate

\[\|u\|_{C^1[D]} \leq C \|\varphi\|_{C^1[0,L]}.\]  

(2.22)

Figure 1. Maximum determinate domain \(D\) of the Cauchy problem

3. Local exact boundary controllability and observability. When system (1.4) possesses no zero eigenvalues (namely, \(l = m\) in (1.6)):

\[\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \ldots, m; s = m + 1, \ldots, n),\]  

(3.1)

the nonlocal boundary conditions (1.8)-(1.9) become

\[v_s(t,0) = G_s(t,v_1(t,0),\ldots,v_m(t,0),v_{m+1}(t,L),\ldots,v_n(t,L)) + H_s(t)\]  

\((s = m + 1,\ldots,n),\)  

(3.2)

\[v_r(t,L) = G_r(t,v_1(t,0),\ldots,v_m(t,0),v_{m+1}(t,L),\ldots,v_n(t,L)) + H_r(t)\]  

\((r = 1,\ldots,m),\)  

(3.3)

where \(v_i (i = 1,\ldots,n)\) are still given by (1.10), and without loss of generality, we assume that

\[G_r(t,0,\ldots,0) \equiv G_s(t,0,\ldots,0) \equiv 0 \quad (r = 1,\ldots,m; s = m + 1,\ldots,n).\]  

(3.4)

Adopting the constructive method given in [10] to establish the exact boundary controllability, we obtain

Theorem 3.1. (Exact boundary controllability) Suppose that \(l_i, \lambda_i, f_i, G_i (i = 1,\ldots,n)\) and \(\varphi\) are all \(C^1\) functions with respect to their arguments. Suppose furthermore that (1.5),(3.1) and (3.4) hold. Let

\[T > L \max_{i=1,\ldots,n} \frac{1}{|\lambda_i(0)|}.\]  

(3.5)

For any given initial data \(\varphi\) and final data \(\psi\), if \(\|\varphi\|_{C^1[0,L]}\) and \(\|\psi\|_{C^1[0,L]}\) are sufficiently small, then there exist boundary controls \(H_i(t) (i = 1,\ldots,n)\) with small \(C^1[0,T]\) norms, such that the corresponding mixed problem (1.4),(1.7) and (3.2)-(3.3) admits a unique semiglobal \(C^1\) solution \(u = u(t,x)\) with small \(C^1\) norm on
the domain $R(T) = \{(t,x)|0 \leq t \leq T, 0 \leq x \leq L\}$, which satisfies exactly the final condition

$$u(T, x) = \psi(x), \quad x \in [0, L].$$

Applying the constructive method given in [8] to establish the exact boundary observability, we have

**Theorem 3.2. (Exact boundary observability)** Suppose that $l_i, \lambda_i, f_i, G_i, H_i$ ($i = 1, \ldots, n$) and $\varphi$ are all $C^1$ functions with respect to their arguments, and $\frac{\partial G_i}{\partial t}(t, \cdot) (i = 1, \ldots, n)$ satisfy local Lipschitz conditions with respect to the variable $v = (v_1, \ldots, v_n)^T$. Suppose furthermore that (1.5), (3.1) and (3.4)-(3.5) hold. Suppose finally that $\|\varphi\|_{C^1[0,L]}$ and $\|H_i\|_{C^1[0,T]} (i = 1, \ldots, n)$ are sufficiently small, and the conditions of $C^1$ compatibility for the mixed problem (1.4), (1.7) and (3.2)-(3.3) are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$. Then the initial data $\varphi$ can be uniquely determined by the boundary observations $\overline{v}_r(t) := v_r(t, 0) (r = 1, \ldots, m)$ and $\overline{v}_s(t) := v_s(t, L) (s = m + 1, \ldots, n)$ together with the known boundary functions $H_i(t)$ ($i = 1, \ldots, n$). Moreover, the following observability estimate holds:

$$\|\varphi\|_{C^1[0,L]} \leq C \left( \sum_{r=1}^{m} \|\overline{v}_r\|_{C^1[0,T]} + \sum_{s=m+1}^{n} \|\overline{v}_s\|_{C^1[0,T]} + \sum_{i=1}^{n} \|H_i\|_{C^1[0,T]} \right),$$

where $C$ is a positive constant possibly depending on $T$.

4. **Application 1—Saint-Venant system.** Consider the Saint-Venant system for a horizontal and cylindrical canal (see [4, 5, 6, 13])

$$\begin{cases}
A_t + (AV)_x = 0, \\
V_t + S_x = 0,
\end{cases} 
$$

(4.1)

where $A > 0$ stands for the area of the cross section occupied by the water, $V$ is the average velocity over the cross section and

$$S = \frac{1}{2}V^2 + gH(A) + gY,$$

(4.2)

where $g$ is the gravity constant, $Y$ is the altitude of the canal bed (we may assume $Y = 0$ without loss of generality), $H$ is the depth of water, which is a $C^1$ function of $A$, satisfying

$$H'(A) > 0, \quad \forall A > 0.$$

(4.3)

Let the initial condition be

$$A(0, x) = A_0(x), \quad V(0, x) = V_0(x), \quad x \in [0, L],$$

(4.4)

and the boundary conditions take the following nonlocal form:

$$S(t, 0) - S(t, L) = h(t),$$

(4.5)

$$Q(t, 0) - Q(t, L) = \overline{q}(t),$$

(4.6)

where $Q = AV$ denotes the flux.

We discuss system (4.1) near a constant subcritical equilibrium $(\tilde{A}, \tilde{V})$ ($\tilde{A} > 0$) which, by definition, satisfies

$$\tilde{V}^2 < g\tilde{A}H'(\tilde{A}).$$

(4.7)

Introducing Riemann Invariants

$$r = \frac{1}{2}(V - \tilde{V} - G(A)), \quad s = \frac{1}{2}(V - \tilde{V} + G(A)),$$

(4.8)
where

\[ G(A) = \int_{\bar{A}}^A \sqrt{gH'(A)} \, dA, \]  

denotes the inverse function of \( G \). By \( (4.7) \), in a \( C^1 \) neighbourhood of \((A, V) = (\bar{A}, \bar{V})\) (correspondingly, \((r, s) = (0, 0)\)), \( (4.1) \) can be equivalently rewritten as

\[
\begin{aligned}
\left\{ \begin{array}{l}
\rho_t + \lambda_1 \, \rho_x = 0, \\
\sigma_t + \lambda_2 \, \sigma_x = 0,
\end{array} \right.
\end{aligned}
\]  

\[ (4.11) \]

where

\[ \lambda_1 = V - \sqrt{gAH'(A)} < 0 < \lambda_2 = V + \sqrt{gAH'(A)}. \]  

(4.12)

The initial condition \((4.4)\) becomes

\[ r(0, x) = r_0(x), \quad s(0, x) = s_0(x), \quad x \in [0, L], \]  

(4.13)

where

\[ r_0(x) = \frac{1}{2}(V_0(x) - \bar{V} - G(A_0(x))), \quad s_0(x) = \frac{1}{2}(V_0(x) - \bar{V} + G(A_0(x))). \]  

(4.14)

In order to change nonlocal boundary conditions \((4.5)-(4.6)\) into the form of \((1.8)-(1.9)\), we first rewrite them as

\[ P_1 := \frac{1}{2}(V_1^2 - V_2^2) + g(H(A_1) - H(A_2)) - h(t) = 0, \]  

(4.15)

\[ P_2 := A_1 V_1 - A_2 V_2 - \mathcal{H}(t) = 0, \]  

(4.16)

where

\[ V_1 = V(t, 0), V_2 = V(t, L), A_1 = A(t, 0), A_2 = A(t, L). \]  

(4.17)

Let

\[ r_1 = r(t, 0), r_2 = r(t, L), s_1 = s(t, 0), s_2 = s(t, L). \]  

(4.18)

Then, at the point \((A, V) = (\bar{A}, \bar{V})\) (correspondingly, \((r, s) = (0, 0)\)),

\[
\det \left( \frac{\partial (P_1, P_2)}{\partial (s_1, r_2)} \right) = 2 \sqrt{\frac{\bar{A}}{gH'(\bar{A})}} \cdot (\bar{V}^2 - g\bar{A}H'(\bar{A})) < 0. \]  

(4.19)

By the Implicit Function Theorem, in a \( C^1 \) neighbourhood of \((A, V) = (\bar{A}, \bar{V})\) (correspondingly, \((r, s) = (0, 0)\)), boundary conditions \((4.5)-(4.6)\) can be furthermore rewritten as

\[ s(t, 0) = F(t, r(t, 0), s(t, L)) + f(t), \]  

(4.20)

\[ r(t, L) = F(t, r(t, 0), s(t, L)) + f(t), \]  

(4.21)

where \( F, \mathcal{F} \) are \( C^1 \) functions with respect to their arguments, and, without loss of generality, we may assume that

\[ F(t, 0, 0) \equiv \mathcal{F}(t, 0, 0) \equiv 0, \]  

(4.22)

consequently,

\[ \| (h, \mathcal{H}) ((C^1([0, T]))^2 \to 0 \iff \| (f, \mathcal{F}) ((C^1([0, T]))^2 \to 0. \]  

(4.23)

Applying Theorem 2.1 to the mixed problem \((4.11), (4.13)\) and \((4.20)-(4.21)\), we obtain
Theorem 4.1. (Semiglobal $C^1$ solution) Let $(\tilde{A},\tilde{V}) (\tilde{A} > 0)$ be a constant subcritical equilibrium. For any preassigned and possibly quite large $T > 0$, if $\|(A_0 - \tilde{A},V_0 - \tilde{V})\|_{C^1([0,L])^2}$ and $\|(h,\tau)\|_{C^1([0,T])^2}$ are sufficiently small, and the conditions of $C^1$ compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, then the mixed problem (4.1) and (4.4)-(4.6) admits a unique semiglobal $C^1$ solution $(A,V) = (A(t,x),V(t,x))$ on $R(T) = \{(t,x)|0 \leq t \leq T, 0 \leq x \leq L\}$, $\|(A - \tilde{A},V - \tilde{V})\|_{C^1([R(T)])^2}$ being small, and the following estimate holds:

$$\|(A - \tilde{A},V - \tilde{V})\|_{C^1([R(T)])^2} \leq C(\|(A_0 - \tilde{A},V_0 - \tilde{V})\|_{C^1([0,L])^2} + \|(h,\tau)\|_{C^1([0,T])^2}),$$

(4.24)

where $C$ is a positive constant possibly depending on $T$.

As in [6], by Theorem 3.1 we get

Theorem 4.2. (Exact boundary controllability) Let $(\tilde{A},\tilde{V}) (\tilde{A} > 0)$ be a constant subcritical equilibrium. Let

$$T > L \max \left\{ \frac{1}{|\lambda_1|}, \frac{1}{\lambda_2} \right\},$$

(4.25)

where

$$\tilde{\lambda}_1 = \tilde{V} - \sqrt{g\tilde{A}H'(\tilde{A})} < 0 < \tilde{\lambda}_2 = \tilde{V} + \sqrt{g\tilde{A}H'(\tilde{A})}.$$  

(4.26)

For any given initial data $(A_0,V_0)$ and final data $(A_T,V_T)$, if $\|(A_0 - \tilde{A},V_0 - \tilde{V})\|_{C^1([0,L])^2}$ and $\|(A_T - \tilde{A},V_T - \tilde{V})\|_{C^1([0,L])^2}$ are sufficiently small (possibly depending on $T$), there exist boundary controls $(h(t),\tau(t))$ with small $\|(h,\tau)\|_{C^1([0,T])^2}$, such that the mixed problem (4.1) and (4.4)-(4.6) admits a unique semiglobal $C^1$ solution $(A,V) = (A(t,x),V(t,x))$ with small $\|(A - \tilde{A},V - \tilde{V})\|_{C^1([R(T)])^2}$ on $R(T)$, which satisfies exactly the final condition:

$$A(T,x) = A_T(x), V(T,x) = V_T(x), \quad x \in [0,L].$$

(4.27)

As in [2], by Theorem 3.2 we obtain

Theorem 4.3. (Exact boundary observability) Let $(\tilde{A},\tilde{V}) (\tilde{A} > 0)$ be a constant subcritical equilibrium and $T$ satisfy (4.25). If $\|(A_0 - \tilde{A},V_0 - \tilde{V})\|_{C^1([0,L])^2}$ and $\|(h,\tau)\|_{C^1([0,T])^2}$ are sufficiently small, and the conditions of $C^1$ compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0,L)$, then the initial data $(A_0,V_0)$ can be uniquely determined by the boundary observation $(\overline{A}(t),\overline{V}(t)) := (A(t,0),V(t,0))$ together with the known boundary functions $(h(t),\tau(t))$. Moreover, the following observability estimate holds:

$$\|(A_0 - \tilde{A},V_0 - \tilde{V})\|_{C^1([0,L])^2} \leq C(\|(\overline{A} - \tilde{A},\overline{V} - \tilde{V})\|_{C^1([0,T])^2} + \|(h,\tau)\|_{C^1([0,T])^2}),$$

(4.28)

where $C$ is a positive constant possibly depending on $T$.

Remark 4.1. Theorem 4.3 still holds if we take the boundary observations $(\overline{A}(t),\overline{V}(t)) := (A(t,L),V(t,L))$ instead of $(A(t,0),V(t,0))$. In fact, the exact boundary observability can be realized as long as the values $(A(t,0),V(t,0),A(t,L),V(t,L))$ or $(r(t,0),s(t,0),r(t,L),s(t,L))$ can be uniquely determined from the boundary observations together with boundary conditions (4.5)-(4.6). For instance, if the boundary observations are taken as $(\overline{S}(t),\overline{Q}(t)) := (S(t,0),Q(t,0))$ or $(\overline{S}(t,L),$
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the exact boundary observability can be also realized with the following observability estimate:

\[ \| (A_0 - \tilde{A}, V_0 - \tilde{V}) \|_{(C^1[0,L])^2} \leq C (\| (S - \tilde{S}, Q - \tilde{Q}) \|_{(C^1[0,T])^2} + \| (h, \tilde{h}) \|_{(C^1[0,T])^2}) \]  

(4.29)

where

\[ \tilde{S} = \frac{1}{2} \tilde{V}^2 + gH(\tilde{A}), \quad \tilde{Q} = \tilde{A}\tilde{V}. \]  

(4.30)

**Remark 4.2.** If the energy type boundary condition (4.5) is replaced by the water level boundary condition

\[ H(A(t,0)) - H(A(t, L)) = h(t), \]  

(4.31)

Theorems 4.1-4.3 still hold.

5. Application 2 — 1-D quasilinear wave equation. Consider the following 1-D quasilinear wave equation

\[ u_{tt} - (K(u, u_x))_x = F(u, u_x, u_t), \]  

(5.1)

where \( K \) is a \( C^2 \) function with

\[ K_v(u, v) > 0 \]  

(5.2)

and \( F \) is a \( C^1 \) function with

\[ F(0, 0, 0) = 0. \]  

(5.3)

By (5.3), \( u \equiv 0 \) is an equilibrium of (5.1). All the discussions in this section will be in a \( C^1 \) neighbourhood of \( (u, u_x, u_t) = (0, 0, 0) \).

Let the initial condition be

\[ u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x), \quad x \in [0, L] \]  

(5.4)

and the boundary conditions take the following nonlocal form:

\[ u(t, 0) - u(t, L) = h(t), \]  

\[ u_x(t, 0) - u_x(t, L) = h(t). \]  

(5.5)

(5.6)

In particular, if \( (h(t), \tilde{h}(t)) \equiv (0, 0), (5.5)-(5.6) \) become the usual periodic boundary conditions.

Reducing the mixed problem (5.1) and (5.4)-(5.6) to a quasilinear hyperbolic system with boundary conditions in the form of (1.8)-(1.9), we will establish the theory of the semiglobal \( C^2 \) solution and then the local exact boundary controllability and observability.

Let

\[ v = u_x, \quad w = u_t \]  

(5.7)

and

\[ U = (u, v, w)^T. \]  

(5.8)

(5.1) can be rewritten to the following first order quasilinear hyperbolic system

\[ \begin{cases} 
  u_t = w, \\
  v_t - w_x = 0, \\
  w_t - K_v(u, v) v_x = \bar{F}(u, v, w) := F(u, v, w) + K_u(u, v) v
\end{cases} \]  

(5.9)

with

\[ \bar{F}(0, 0, 0) = 0. \]  

(5.10)
By (5.2), (5.9) is a strictly hyperbolic system with three distinct real eigenvalues
\[ \lambda_1(U) = -\sqrt{K_v(u,v)} < \lambda_2(U) \equiv 0 < \lambda_3(U) = \sqrt{K_v(u,v)} \]  
and a complete set of left eigenvectors
\[ l_1(U) = (0, \sqrt{K_v(u,v)}, 1), \quad l_2(U) = (1, 0, 0), \quad l_3(U) = (0, -\sqrt{K_v(u,v)}, 1). \]  
The initial condition correspondingly becomes
\[ U(0, x) = (\varphi(x), \varphi'(x), \psi(x))^T, \quad x \in [0, L]. \]  
Let
\[ V_i = l_i(U)U \quad (i = 1, 2, 3), \]  
i.e.,
\[ V_1 = \sqrt{K_v(u,v)} v + w, \quad V_2 = u, \quad V_3 = -\sqrt{K_v(u,v)} v + w. \]  
At the point \( U = 0 \), we have
\[ \frac{\partial(V_1, V_2, V_3)}{\partial(u, v, w)} = \begin{pmatrix} 0 & \sqrt{K_v(0,0)} & 1 \\ 1 & 0 & 0 \\ 0 & -\sqrt{K_v(0,0)} & 1 \end{pmatrix}, \]  
then
\[ \frac{\partial(u, v, w)}{\partial(V_1, V_2, V_3)} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}. \]  
Noting the condition of \( C^0 \) compatibility at the points \( (t, x) = (0, 0) \) and \( (0, L) \):
\[ \varphi(0) - \varphi(L) = h(0), \]  
the boundary condition (5.5) is equivalent to
\[ w(t, 0) - w(t, L) = h'(t). \]  
In order to reduce (5.6) and (5.19) into the form of (1.8)-(1.9), we first rewrite them to
\[ P_1 := w(t, 0) - w(t, L) - h'(t) = 0, \]  
\[ P_2 := v(t, 0) - v(t, L) - h(t) = 0. \]  
Let
\[ w_1 = V_3(t, 0), \quad w_2 = V_1(t, L). \]  
At the point of \( U = 0 \), by (5.17) it is easy to see that
\[ \det \left( \frac{\partial(P_1, P_2)}{\partial(w_1, w_2)} \right) = -\frac{1}{2\sqrt{K_v(0,0)}} < 0, \]  
then, in a \( C^0 \) neighbourhood of \( U = 0 \), (5.6) and (5.19) can be equivalently rewritten as
\[ V_3(t, 0) = G_3(t, V_1(t, 0), V_2(t, 0), V_3(t, L), V_3(t, L)) + H_3(t), \]  
\[ V_1(t, L) = G_1(t, V_1(t, 0), V_2(t, 0), V_2(t, L), V_3(t, L)) + H_1(t), \]  
where \( G_1, G_3 \) are \( C^1 \) functions with respect to their arguments and satisfy
\[ G_1(t, 0, 0, 0, 0) \equiv G_3(t, 0, 0, 0, 0) \equiv 0, \]  
consequently,
\[ \|(h', \overline{\eta})\|_{(C^1[0,T])^2} \to 0 \iff \|(H_1, H_3)\|_{(C^1[0,T])^2} \to 0. \]
As in [11] (or [15]), applying Theorem 2.1 to the mixed problem (5.9), (5.13) and
(5.24)-(5.25), we obtain

**Theorem 5.1. (Semiglobal C² solution)** For any preassigned and possibly quite large
$T > 0$, if $\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]}$ and $\|(h, \overline{h})\|_{C^2[0,T] \times C^1[0,T]}$ are sufficiently small (possibly depending on $T$), and the conditions of $C^2$ compatibility are satisfied at
the points $(t, x) = (0, 0)$ and $(0, L)$, then the mixed problem (5.1) and (5.4)-(5.6)
admits a unique semiglobal $C^2$ solution $u = u(t, x)$ with small $C^2$ norm on the
domain $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ and the following estimate holds:

$$
\|u\|_{C^2[R(T)]} \leq C(\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]} + \|(h, \overline{h})\|_{C^2[0,T] \times C^1[0,T]}),
$$

where $C$ is a positive constant possibly depending on $T$.

Based on Theorem 5.1, adopting a similar constructive method as in [11] (or
[15]), we obtain immediately

**Theorem 5.2. (Exact boundary controllability)** Let

$$
T > \frac{L}{\sqrt{K_0(0, 0)}},
$$

(5.29)

For any given initial data $(\varphi, \psi)$ and final data $(\Phi, \Psi)$, if the norms

$$
\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]} \quad \text{and} \quad \|(\Phi, \Psi)\|_{C^2[0,L] \times C^1[0,L]}
$$

are sufficiently small, then there exist boundary controls $(h(t), \overline{h}(t))$ with small
$\|(h, \overline{h})\|_{C^2[0,T] \times C^1[0,T]}$, such that the mixed problem (5.1) and (5.4)-(5.6)
adopts a unique $C^2$ solution $u = u(t, x)$ with small $C^2$ norm on $R(T)$, which satisfies exactly
the final condition

$$
u(T, x) = \Phi(x), \quad u_t(T, x) = \Psi(x), \quad x \in [0, L].
$$

(5.30)

By the constructive method in [7] (or [3]), we get

**Theorem 5.3. (Exact boundary observability)** Let $T$ satisfy (5.29). If

$$
\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]} \quad \text{and} \quad \|(h, \overline{h})\|_{C^2[0,T] \times C^1[0,T]}
$$

are sufficiently small, and the conditions of $C^2$ compatibility are satisfied at
the points $(t, x) = (0, 0)$ and $(0, L)$, then the initial data $(\varphi, \psi)$ can be uniquely deter-
mined by the boundary observations $(\overline{u}(t), \overline{\psi}(t)) := (u(t, 0), u_x(t, 0))$ together with
the boundary functions $(h(t), \overline{h}(t))$. Moreover, the following observability estimate
holds:

$$
\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]} \leq C(\|(\overline{u}, \overline{\psi})\|_{C^2[0,T] \times C^1[0,T]} + \|(h, \overline{h})\|_{C^2[0,T] \times C^1[0,T]}),
$$

(5.31)

where $C$ is a positive constant possibly depending on $T$.

**Remark 5.1.** If the boundary observations $(\overline{u}(t), \overline{\psi}(t))$ are taken as $(u(t, 0), u_x(t, L))$
or $(u(t, L), u_x(t, L))$ or $(u(t, L), u_x(t, 0))$ instead of $(u(t, 0), u_x(t, 0))$, Theorem 5.3
still holds. In fact, the exact boundary observability always holds if $(u(t, 0), u_x(t, 0),
\overline{u}(t, L), u_x(t, L))$ can be uniquely determined by the boundary observations and
boundary conditions (5.5)-(5.6).

6. Exact boundary controllability for a system in a network with loop can not be realized generically.
In this section we give some examples to show that, generically speaking, the number of both boundary controls and boundary observations can not be reduced and then the exact boundary controllability for a hyperbolic system in a network with loop can not be realized.
6.1. Linearized Saint-Venant system. For the linearized Saint-Venant system near a constant subcritical equilibrium \((\bar{A}, \bar{V}) (\bar{A} > 0)\)

\[
\frac{\partial}{\partial t} \begin{pmatrix} A \\ V \end{pmatrix} + \begin{pmatrix} \bar{V} \\ gH'(\bar{A}) \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} A \\ V \end{pmatrix} = 0,
\]

we consider the following nonlocal boundary conditions (cf. (4.31) and (4.6)):

\[
A(t, L) - A(t, 0) = 0
\]

and

\[
V(t, L) - V(t, 0) = h(t),
\]

which correspond to a loop.

The two eigenvalues and the corresponding left eigenvectors are given by

\[
\lambda_1 = \bar{V} - \sqrt{g\bar{A}H'(\bar{A})} < 0 < \lambda_2 = \bar{V} + \sqrt{g\bar{A}H'(\bar{A})}
\]

and

\[
l_1 = (\sqrt{g\bar{A}H'(\bar{A})}, -\bar{A}), \quad l_2 = (\sqrt{g\bar{A}H'(\bar{A})}, \bar{A}),
\]

respectively. Using the Riemann invariants

\[
\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \sqrt{g\bar{A}H'(\bar{A})} \\ \sqrt{g\bar{A}H'(\bar{A})} \end{pmatrix} \begin{pmatrix} A \\ V \end{pmatrix},
\]

system (6.1) can be rewritten into the following diagonal form

\[
\begin{cases}
\frac{\partial r}{\partial t} + \lambda_1 \frac{\partial r}{\partial x} = 0, \\
\frac{\partial s}{\partial t} + \lambda_2 \frac{\partial s}{\partial x} = 0,
\end{cases}
\]

and (6.2)-(6.3) are equivalently transformed into the following boundary conditions:

\[
r(t, L) - r(t, 0) = -\bar{A}h(t)
\]

and

\[
s(t, L) - s(t, 0) = \bar{A}h(t).
\]

For the control problem, there are formally two controls in (6.8)-(6.9), but they are not independent. We will show that system (6.7)-(6.9) is not exactly controllable by means of \(h(t)\).

Let \((r_0, s_0)\) be a constant initial data satisfying

\[
 r_0 + s_0 > 0.
\]

It is easy to see that the conditions of \(C^1\) compatibility are satisfied at the point \((t, x) = (0, 0)\) and \((0, L)\). Assume that there exists a control \(h \in C^1[0, T]\), such that system (6.7)-(6.9) with the initial data \((r_0, s_0)\) admits a unique \(C^1\) solution \((r(t, x), s(t, x))\) on the domain \(R(T) = \{(t, x) | 0 \leq t \leq T, \ 0 \leq x \leq L\}\), which satisfies the final conditions

\[
r(T, x) = s(T, x) = 0, \quad 0 \leq x \leq L.
\]

Then, integrating (6.7) on \(R(T)\) yields

\[
\begin{cases}
r_0L + \lambda_1 \bar{A} \int_0^T h(t)dt = 0, \\
s_0L - \lambda_2 \bar{A} \int_0^T h(t)dt = 0,
\end{cases}
\]
hence
\[ \lambda_2 r_0 + \lambda_1 s_0 = 0. \]  
(6.13)
Specially taking
\[ (r_0, s_0) = (\alpha \lambda_2, \lambda_1), \]  
(6.14)
where \( \alpha \) is a positive constant such that
\[ r_0 + s_0 = \alpha \lambda_2 + \lambda_1 > 0 \iff \alpha > \frac{\sqrt{g A H'(\bar{A}) - \bar{V}}}{\sqrt{g A H'(\bar{A}) + \bar{V}}}. \]  
(6.15)
we get a contradiction
\[ \lambda_1^2 + \alpha \lambda_2^2 = 0. \]  
(6.16)

6.2. 1-D linear wave equation. First we show that the number of boundary observations in Theorem 5.3 cannot be reduced. For this purpose, consider the following mixed problem for the linear wave equation with the periodic boundary conditions:

\[
\begin{cases}
\phi_{tt} - \phi_{xx} = 0, \\
\phi(t, 0) = \phi(t, 2\pi), \\
\phi_x(t, 0) = \phi_x(t, 2\pi), \\
\phi(0, x) = \phi_0(x), \quad \phi_t(0, x) = \phi_1(x), \quad x \in [0, 2\pi].
\end{cases}
\]  
(6.17)-(6.20)

By Theorem 5.3, if the boundary observations are chosen as \((\phi(0, x), \phi_x(0, x))\) and \(T \geq 2\pi\), the exact boundary observability for (6.17)-(6.20) holds on the time interval \([0, T]\). However, if the boundary observation is only \(\phi(0, x)\) (resp., \(\phi_x(0, x)\)), the exact boundary observability for (6.17)-(6.20) cannot be realized on any time interval \([0, T]\) \(T > 0\). To show this, it suffices to find a nontrivial solution to (6.17)-(6.20), such that the boundary observation \(\phi(t, 0)\) (resp., \(\phi_x(t, 0)\)) is identically equal to zero, while the initial data \((\phi_0(x), \phi_1(x))\) is not identically zero. In fact,
\[ \phi(t, x) = \sin nt \sin nx, \quad n \in \mathbb{Z}^+ \]  
(6.21)
satisfies (6.17)-(6.20) with \((\phi_0(x), \phi_1(x)) \equiv (0, n \sin nx)\) and \(\phi(t, 0) \equiv 0\). Therefore, observing only \(\phi(t, 0)\) is not sufficient to guarantee the exact boundary observability. Similarly,
\[ \phi(t, x) = \cos nt \cos nx, \quad n \in \mathbb{Z}^+ \]  
(6.22)
satisfies (6.17)-(6.20) with \((\phi_0(x), \phi_1(x)) \equiv (\cos nx, 0)\) and \(\phi_x(t, 0) \equiv 0\). Then, observing only \(\phi_x(t, 0)\) is not sufficient to guarantee the exact boundary observability, either.

We now show that the number of boundary controls in Theorem 5.2 cannot be reduced. For this purpose, we first suppose that there exist \(T > 0\) and a boundary control \(\tilde{h}(t)\) such that the solution \(y = y(t, x)\) to the following control system

\[
\begin{cases}
y_{tt} - y_{xx} = 0, \\
y(t, 0) = y(t, 2\pi), \\
y_x(t, 0) = y_x(t, 2\pi) + \tilde{h}(t), \\
y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), \quad x \in [0, 2\pi]
\end{cases}
\]  
(6.23)-(6.26)
satisfies exactly the final null condition
\[ y(T, x) \equiv y_t(T, x) \equiv 0, \quad x \in [0, 2\pi]. \]  
(6.27)
Multiplying the wave equation (6.23) by the solution $\phi = \phi(t, x)$ to system (6.17)-(6.20), and then integrating on $[0, T] \times [0, 2\pi]$, we obtain
\[
\int_0^T \int_0^{2\pi} y_t(t, x)\phi(t, x)dxdt = \int_0^T \int_0^{2\pi} y_{xx}(t, x)\phi(t, x)dxdt. \tag{6.28}
\]
By integration by parts and using (6.17)-(6.20) and (6.24)-(6.27) it follows that
\[
\int_0^{2\pi} (-y_1(x)\phi_0(x) + y_0(x)\phi_1(x))dx = -\int_0^T \tilde{h}(t)\phi(t, 2\pi)dt. \tag{6.29}
\]
In particular, taking the initial data in (6.26) to be
\[
y_0(x) = \sin nx, \quad y_1(x) \equiv 0, \quad x \in [0, 2\pi] \tag{6.30}
\]
and $\phi(t, x)$ to be given by (6.21), from (6.29) we get a contradiction
\[
n\int_0^{2\pi} \sin^2 nx dx = 0. \tag{6.31}
\]

**Remark 6.1.** Noting (6.24), we conclude from the above that: the exact boundary controllability for a system in a network with loop cannot be realized generically.

Similarly, it can be shown that if the initial data in (6.26) is taken as
\[
y_0(x) \equiv 0, \quad y_1(x) = \cos nx, \quad x \in [0, 2\pi], \tag{6.32}
\]
there do not exist $T > 0$ and a boundary control $h(t)$ such that the solution $y = y(t, x)$ to the following control system
\[
\begin{aligned}
y_{tt} - y_{xx} &= 0, \\
y(t, 0) &= y(t, 2\pi) + h(t), \\
y_x(t, 0) &= y_x(t, 2\pi), \\
y(0, x) &= y_0(x), \quad y_t(0, x) = y_1(x), \quad x \in [0, 2\pi]
\end{aligned} \tag{6.33}
\]
satisfies exactly the null final condition (6.27).

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