CONTROL OF A SCALAR CONSERVATION LAW WITH A NONLOCAL VELOCITY

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Abstract. In this paper, we survey some results on control problems for a scalar conservation law with a nonlocal velocity, that models a highly re-entrant manufacturing system in semi-conductor production. For the open-loop system, we show the results on an optimal control problem, state controllability and out-flux controllability. While for the closed-loop system, we prove, by using a Lyapunov function approach, the exponential stabilization results in certain cases.

1. Introduction. This paper studies some control problems for control systems governed by the following scalar hyperbolic conservation law with a nonlocal velocity:

\[ \partial_t \rho(t,x) + \partial_x (\rho(t,x)\lambda(W(t))) = 0 \quad \text{where} \quad W(t) = \int_0^1 \rho(t,x) \, dx, \]

on a rectangular domain \((0,T) \times (0,1)\) or the semi-infinite strip \((0,\infty) \times (0,1)\). We assume that the velocity function \(\lambda\) is in \(C^1(R; (0, +\infty))\). Let us recall that the special case

\[ \lambda(s) = \frac{1}{1+s}, \quad s \in [0, +\infty), \]

was, for example, used in [5, 17]. Compared to the results in [12, 13, 14], the non-negativeness of the data is eliminated in the whole paper, since \(\lambda\) is defined on \(R\) (with suitable extension if necessary) instead of \([0, +\infty)\).

In the manufacture system, the initial data is given as the product density:

\[ \rho(0, x) = \rho_0(x), \quad x \in (0,1). \]

For the open-loop system, the control is acted on the influx \(u(t) := \rho(t,0)\lambda(W(t)):\)

\[ u(t) = h(t). \]

While for the closed-loop control system, the following output feedback law is used:

\[ u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \]

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in which the output/measurement is the out-flux $y(t) := \rho(t,1)\lambda(W(t))$, $k \in \mathbb{R}$ is a tuning parameter, $\rho \in \mathbb{R}$ is the equilibrium that we want to stabilize as $t \to +\infty$.

The conservation law (1) arises in modeling of the semiconductor manufacturing systems, see e.g. [5, 17]. These systems are characterized by their highly re-entrant feature with very high volume (number of parts manufactured per unit time) and very large number of consecutive production steps as well. The main character of this model is described in terms of the velocity function $\lambda$ which is a function of the total quantity of products in the plant(s).

The control problems for conservation laws and general hyperbolic systems have been widely studied. As for the controllability of nonlinear hyperbolic equations, we refer to [8, 15, 19] for solutions without shocks, and to [1, 2, 3, 4, 6, 16, 20] for solutions with shocks. As for asymptotic stability/stabilization of hyperbolic systems, we refer to [18, 21] for the strategy of careful analysis of solutions along the characteristic curves, and to [7, 9, 10, 23, 24] for the Lyapunov function approach.

Concerning the manufacturing model of (1) itself, an optimal control problem, motivated by [5, 17], related to the Demand Tracking Problem was studied in [12] (see also [22] for a generalized system where $\lambda = \lambda(x,W(t))$). Later, the state-controllability and out-flux controllability for the open-loop system have been obtained in [13]. We show these results without proof in Section 3.

In Section 4, we prove the exponential stabilization for the closed-loop system (under the feedback law (4)) by Lyapunov function approach. The Lyapunov functions that we construct are inspired by [7, 11, 23] with necessary modifications according to the nonlocal feature of the nonlinear system, see [14]. The stabilization results depend on the equilibrium $\overline{\rho} \neq 0$ that we want to stabilize and the velocity function $\lambda$ through the following quantity

$$d := \frac{\overline{\rho}'(\overline{\rho})}{\lambda(\overline{\rho})}.$$  (5)

2. Well-posedness of Cauchy problems. As for the open-loop system, we recall first the definition of the weak solution to the Cauchy problem (see, e.g., [9, Section 2.1]) and then its well-posedness; see [12, 22] for its proof.

**Definition 2.1.** Let $T > 0$, $p \in [1, +\infty)$, $\rho_0 \in L^p(0,1)$ and $h \in L^p(0,T)$ be given. A weak solution of the Cauchy problem

$$\begin{cases}
\rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, & t \in (0,T), x \in (0,1), \\
\rho(0,x) = \rho_0(x), & x \in (0,1), \\
u(t) = h(t), & t \in (0,T)
\end{cases}$$  (6)

is a function $\rho \in C^0([0,T];L^p(0,1))$ such that, for every $\tau \in [0,T]$ and every $\varphi \in C^1([0,\tau] \times [0,1])$ satisfying $\varphi(\tau,x) = 0, \forall x \in [0,1]$ and $\varphi(t,1) = 0, \forall t \in [0,\tau]$, one has

$$\int_0^\tau \int_0^1 \rho(t,x)(\varphi_x(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt + \int_0^\tau h(t)\varphi(t,0)dt + \int_0^1 \rho_0(x)\varphi(0,x)dx = 0.$$  (7)

**Theorem 2.2 (Well-posedness of open-loop system).** Let $T > 0$ and $p \in [1, +\infty)$ be given. For any $\rho_0 \in L^p(0,1)$ and $h \in L^p(0,T)$, the Cauchy problem
admits a unique weak solution $\rho \in C^0([0, T]; L^p(0, 1))$, and additionally $\rho \in C^0([0, 1]; L^p(0, T))$. Moreover, the following maps
\begin{align*}
(\rho_0, h) \in L^p(0, 1) \times L^p(0, T) &\mapsto \rho \in C^0([0, T]; L^p(0, 1)), \\
(\rho_0, h) \in L^p(0, 1) \times L^p(0, T) &\mapsto y \in L^p(0, T)
\end{align*}
are continuous.

**Remark 1.** Theorem 2.2 has been generalized to the case where $\lambda = \lambda(x, W(t))$ in [22].

As for the closed-loop system, the definition and the well-posedness of the weak solution to the Cauchy problem are as follows.

**Definition 2.3.** Let $\overline{\varphi} \in \mathbb{R}, p \in [1, +\infty), k \in \mathbb{R}$ and $\rho_0 \in L^p(0, 1)$ be given. A weak solution of the Cauchy problem
\begin{equation}
\begin{cases}
\rho_t(t, x) + (\rho(t, x))\lambda(W(t)))_x = 0, & t \in (0, +\infty), x \in (0, 1), \\
\rho(0, x) = \rho_0(x), & x \in (0, 1), \\
\rho(t) - \overline{\varphi}\lambda(\overline{\varphi}) = k(y(t) - \overline{\varphi}\lambda(\overline{\varphi})), & t \in (0, +\infty)
\end{cases}
\end{equation}
is a function $\rho \in C^0([0, +\infty); L^p(0, 1))$ such that, for every $T > 0$, every $\tau \in [0, T]$ and every $\varphi \in C^1([0, \tau] \times [0, 1])$ satisfying $\varphi(\tau, x) = 0, \forall x \in [0, 1]$ and $\varphi(t, 1) = 0, \forall t \in [0, \tau]$, one has
\begin{equation}
\begin{aligned}
&-\int_0^1 \int_0^1 \rho(t, x)\varphi_t(t, x) + \lambda(W(t))\varphi_x(t, x)\, dx
dt - \int_0^1 \rho_0(x)\varphi(0, x)\, dx \\
&\quad + \int_0^T (y(t)\varphi(t, 1) - [ky(t) + (1 - k)\overline{\varphi}\lambda(\overline{\varphi})]\varphi(t, 0))\, dt = 0.
\end{aligned}
\end{equation}

**Theorem 2.4 (Well-posedness of closed-loop system).** Let $\overline{\varphi} \in \mathbb{R}, p \in [1, +\infty)$ and $k \in \mathbb{R}$ be given. For any given $\rho_0 \in L^p(0, 1)$, the Cauchy problem (10) has a unique weak solution $\rho \in C^0([0, +\infty); L^p(0, 1))$. Moreover, for every $T > 0$, the following maps
\begin{align}
\rho_0 &\in L^p(0, 1) \mapsto \rho \in C^0([0, T]; L^p(0, 1)), \\
\rho_0 &\in L^p(0, 1) \mapsto (u, y) \in L^p(0, T) \times L^p(0, T),
\end{align}
are continuous.

**Sketch of proof.** We introduce the characteristic curve:
\begin{equation}
\frac{d\xi}{ds} = \lambda(W(s)), \quad s \geq 0 \quad \text{with} \quad W(s) = \int_0^1 \rho(s, x)\, dx.
\end{equation}
Since $\rho$ is constant along the characteristics, we define a solution candidate in terms of $\xi$:
\begin{equation}
\rho(t, x) = \begin{cases}
\rho_0(x - \xi(t)), & \text{if } 0 \leq \xi(t) \leq x \leq 1, \\
kp(\xi^{-1}(\xi(t) - x), 1) + \frac{(1 - k)\overline{\varphi}\lambda(\overline{\varphi})}{\lambda(W(\xi^{-1}(\xi(t) - x)))}, & \text{if } 0 \leq x \leq \xi(t) - n + 1 \leq 1, \text{ or } 0 \leq \xi(t) - n \leq x \leq 1 \text{ for } n \in \mathbb{N}.
\end{cases}
\end{equation}
By fixed point argument as in [12], we prove that (15) is indeed the unique weak solution to the closed-loop system (6). Thanks to (15), we also get the continuity of the maps (12) and (13): see the proof of [22, Theorem 4.1].
3. Control problems of the open-loop system. In this section, we give some results on control problems for the open-loop system (6).

First we obtain the existence of the $L^2$-optimal control for the demand tracking problem. For its proof, we refer to [12].

**Theorem 3.1 (Optimal control problem).** Let $T > 0$ be given. For any $\rho_0 \in L^2(0,1)$ and any $y_d \in L^2(0,T)$, let $J$ be the functional on $L^2(0,T)$:

$$J(h) := \int_0^T |h(t)|^2 dt + \int_0^T |y(t) - y_d(t)|^2 dt,$$

where $y(t) = \rho(t,1)\lambda(W(t))$ is the out-flux corresponding to the influx $h \in L^2(0,T)$ and initial data $\rho_0$. Then, there exists $h_\infty \in L^2(0,T)$ such that $J(h_\infty) = \inf_{h \in L^2(0,T)} J(h)$.

Next, we have three theorems on controllability of the system (6); see [13] for proofs.

**Theorem 3.2 (Local state controllability).** Let $\overline{\rho} \in \mathbb{R}$ be the given constant equilibrium and let

$$T_0 := \frac{1}{\lambda(\overline{\rho})}.$$  

Then, for any $T > T_0$, any $\varepsilon > 0$ and any $p \in [1, +\infty)$, there exists $\nu > 0$ such that, for any $\rho_0 \in L^p(0,1)$ and any $\rho_1 \in L^p(0,1)$ with $\|\rho_0(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu$ and $\|\rho_1(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu$, there exists $h \in L^p(0,T)$ with

$$\|h(\cdot) - p\lambda(\overline{\rho})\|_{L^p(0,T)} \leq \varepsilon,$$

such that the weak solution $\rho \in C^0([0,T]; L^p(0,1))$ to the Cauchy problem (6) satisfies the final condition

$$\rho(T, x) = \rho_1(x), \quad x \in (0,1)$$

and the estimate

$$\|\rho(t, \cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \varepsilon, \quad \forall t \in [0,T].$$

**Theorem 3.3 (Global state controllability).** For any $p \in [1, +\infty)$, any $\rho_0 \in L^p(0,1)$ and any $\rho_1 \in L^p(0,1)$, there exists $T_1 > 0$ (depending on $\rho_0$ and $\rho_1$) such that the following holds: For any $T \geq T_1$, there exists $h \in L^p(0,T)$ such that the weak solution $\rho \in C^0([0,T]; L^p(0,1))$ to the Cauchy problem (6) satisfies the final condition (19).

**Theorem 3.4 (Out-flux controllability).** Let $\overline{\rho} \in \mathbb{R}$ be the given constant equilibrium and let $T_0$ be given by (17). For any $p \in [1, +\infty)$, any $\varepsilon > 0$ and any $T_1, T$ with $T_0 < T_1 < T$, there exists $\nu > 0$ such that the following holds: For any $\rho_0 \in L^p(0,1)$ and any $y_d \in L^p(T_1,T)$ with $\|\rho_0(\cdot) - \overline{\rho}\|_{L^p(0,1)} \leq \nu$ and $\|y(\cdot) - p\lambda(\overline{\rho})\|_{L^p(T_1,T)} \leq \nu$, there exists $h \in L^p(0,T)$ satisfying (18) such that the weak solution $\rho \in C^0([0,T]; L^p(0,1))$ to the Cauchy problem (6) satisfies the estimate (20) and the out-flux condition

$$y(t) = y_d(t), \quad t \in (T_1,T).$$
4. Exponential stabilization of the closed-loop system. In this section, we stabilize the nonlinear system to an equilibrium $\overline{\rho} \in \mathbb{R}$ by using Lyapunov function approach. The stabilization results have been divided into two cases: $\overline{\rho} = 0$ and $\overline{\rho} \neq 0$, because the situation of $\overline{\rho} = 0$ is much more complicated than that of $\overline{\rho} \neq 0$. The main assumptions, $d > -1$ and $k \in (-1,1)$, of exponential stability are due to the spectral analysis of the linearized system of (10) near $\overline{\rho}$, see [14] for the details. Based on these facts, we construct the candidate Lyapunov functions for the linearized system, with which we obtain exponential stabilization for the nonlinear system as a perturbation to the linearized one. The proofs are written for regular solutions, but they are true for weak solution as well because of the continuous dependence of the weak solution with respect to the initial data. More precisely, we have the following theorems.

**Theorem 4.1 (Stabilization to $\overline{\rho} = 0$).** Let $k \in (-1,1)$. For every $R > 0$, there exist constants $C = C(k,R) > 0$ and $\alpha = \alpha(k,R) > 0$ such that for any $\rho_0 \in L^2(0,1)$ with

$$\|\rho_0\|_{L^1(0,1)} \leq R,$$

the solution $\rho \in C^0([0,\infty);L^2(0,1))$ to the Cauchy problem

$$\begin{cases}
\rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, & t \in (0,\infty), x \in (0,1),
\rho(0,x) = \rho_0(x), & x \in (0,1),
\rho(t) = k(y(t)), & t \in (0,\infty)
\end{cases}$$

satisfies

$$\|\rho(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\rho_0\|_{L^2(0,1)}, \quad \forall t \in [0,\infty).$$

**Sketch of proof.** Motivated by the Lyapunov functions used in [7, 11, 23], we introduce the following Lyapunov function (equivalent to $\|\rho(t,\cdot)\|_{L^2(0,1)}$

$$L_1(t) := \int_0^t e^{-\beta \cdot x} \rho^2(t,x) dx, \quad \forall t \in [0,\infty),$$

where $\beta > 0$ is a positive constant to be determined. Then, along the trajectories of (23),

$$\dot{L}_1(t) = -\beta \lambda(W(t))L_1(t) + (\lambda(W(t)))^{-1}(k^2 - e^{-\beta})y^2(t) \leq -\beta \lambda(W(t))L_1(t)$$

by letting $\beta > 0$ be such that $k^2 < e^{-\beta} < 1$.

In order to get exponential decay of the solution as $t \to \infty$, it suffices to prove the uniform boundedness of $W(\cdot)$. In fact, $t \mapsto \int_0^t |\rho(t,x)| dx$ is a nonincreasing function which leads to that

$$|W(t)| \leq \|\rho_0\|_{L^1(0,1)} \leq R, \quad \forall t \in [0,\infty).$$

Then, from (22), (26) and (27), we conclude the proof of Theorem 4.1.

**Theorem 4.2 (Stabilization to $\overline{\rho} \neq 0$).** Assume that $d > -1$. Let $k \in (-1,1)$. Then there exist constants $\varepsilon = \varepsilon(\overline{\rho},k) > 0$, $C = C(\overline{\rho},k) > 0$ and $\alpha = \alpha(\overline{\rho},k) > 0$ such that the following holds: For every $\rho_0 \in L^2(0,1)$ with

$$\|\rho_0(\cdot) - \overline{\rho}\|_{L^2(0,1)} \leq \varepsilon,$$

the weak solution $\rho \in C^0([0,\infty);L^2(0,1))$ to the Cauchy problem (10) satisfies

$$\|\rho(t,\cdot) - \overline{\rho}\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\rho_0(\cdot) - \overline{\rho}\|_{L^2(0,1)}, \quad \forall t \in [0,\infty).$$

□
Sketch of proof. The proof is divided into two cases.

Case 1: $|d| < 1$. We construct a Lyapunov function (equivalent to $\|\rho(t, \cdot) - \overline{\rho}\|_{L^2(0,1)}^2$):

$$L_2(t) := \int_0^1 e^{-\beta x} (\rho(t,x) - \overline{\rho})^2 \, dx + a(W(t) - \overline{\rho})^2, \quad \forall t \in [0, +\infty),$$

with

$$a := \frac{e^{-\beta} - k}{1 - k} d, \quad k^2 < e^{-\beta} < 1. \tag{31}$$

Note that

$$\dot{W}(t) = \int_0^1 \rho_t(t,x) \, dx = u(t) - y(t). \tag{32}$$

Note also that

$$\lambda(W(t)) = \lambda(\overline{\rho}) + \lambda'(\overline{\rho})(W(t) - \overline{\rho}) + o(1)(W(t) - \overline{\rho}), \quad \forall t \in [0, +\infty) \tag{33}$$

since $\lambda$ is of class $C^1$. Here and in the following, $o(1)$ denotes various quantities which tend to 0 as $|W(t) - \overline{\rho}| \to 0$. Let us compute the time derivative of $L_2(t)$ for any classical solution of (10):

$$\dot{L}_2(t) = -\lambda(W(t)) \int_0^1 e^{-\beta x} (\rho(t,x) - \overline{\rho})^2 \, dx + 2a(W(t) - \overline{\rho})\dot{W}(t)$$

$$= -\lambda(W(t)) \int_0^1 e^{-\beta x} (\rho(t,x) - \overline{\rho})^2 \, dx + BT, \tag{34}$$

where

$$BT = \frac{(u(t) - \overline{\rho} \lambda(W(t)))^2 - e^{-\beta}(y(t) - \overline{\rho} \lambda(W(t)))^2}{\lambda(W(t))} + 2a(W(t) - \overline{\rho})(u(t) - y(t))$$

$$\leq \lambda(\overline{\rho}) |d^2(1 - e^{-\beta}) + o(1)|(W(t) - \overline{\rho})^2. \tag{35}$$

due to (5), (31), (32) and (33). Then, it follows from (30), (31), (33), (34), (35) and Hölder inequality that for some constant $c > 0$,

$$\dot{L}_2(t) \leq -c\beta \lambda(\overline{\rho})[1 - d^2(e^\beta - 1)^2 e^{-\beta} \beta^{-2} + o(1)]L_2(t). \tag{36}$$

Note that $1 - d^2(e^\beta - 1)^2 e^{-\beta} \beta^{-2} \to 1 - d^2 > 0$ as $\beta \to 0^+$. Finally, we choose firstly $\beta$, then $\varepsilon$, sufficiently small, so that $1 - d^2(e^\beta - 1)^2 e^{-\beta} \beta^{-2} + o(1) > (1 - d^2)/2 > 0$ by the continuity of the mapping $\rho_0 \mapsto \rho$. This concludes the proof of Theorem 4.2 in the case $|d| < 1$.

Case 2: $d \geq 1$. Let

$$V_1(t) := \int_0^1 (\rho(t,x) - \overline{\rho})^2 \, dx + d(W(t) - \overline{\rho})^2. \tag{37}$$

Then, by (32), (33) and the Cauchy-Schwarz inequality,

$$\dot{V}_1(t) = -\lambda(W(t)) \int_0^1 ((\rho(t,x) - \overline{\rho})^2)_x \, dx + 2d(W(t) - \overline{\rho})\dot{W}(t)$$

$$= \lambda(W(t))(k^2 - 1) \xi^2(t,1) + o(1)W(t)\xi(t,1) + o(1)W^2(t)$$

$$\leq \lambda(\overline{\rho})(k^2 - 1 + o(1))\xi^2(t,1) + o(1)W^2(t), \tag{38}$$
where $\xi(t, x) := (\rho(t, x) - \overline{\rho}) + d(W(t) - \overline{W})$. It is easy to check that $\xi$ satisfies the following Cauchy problem
\[
\begin{cases}
\xi_t(t, x) + \lambda(W(t))\xi_x(t, x) = d\dot{W}(t), & t \in (0, +\infty), x \in (0, 1), \\
\xi(0, x) = (\rho_0(x) - \overline{\rho}) + d(W(0) - \overline{W}), & x \in (0, 1), \\
\xi(t, 0) = k\xi(t, 1) + (k - 1) \left(\frac{\overline{\rho}(\lambda(W(t)) - \lambda(\overline{\rho}))}{\lambda(W(t))} - d(W(t) - \overline{W})\right), & t \in (0, +\infty).
\end{cases}
\] (39)

Let
\[
V_2(t) := \int_0^1 e^{-x} \xi^2(t, x) dx.
\] (40)

Then, by (32), (33), (39) and the Cauchy-Schwarz inequality,
\[
\begin{align*}
\dot{V}_2(t) &= -\lambda(W(t)) \int_0^1 e^{-x}(\xi^2(t, x))_x dx + 2d\dot{W}(t) \int_0^1 e^{-x}\xi(t, x) dx \\
&= -\lambda(W(t))V_2(t) + \lambda(W(t))(k^2 - e^{-1})\xi^2(t, 1) + o(1)W(t)\xi(t, 1) + o(1)W^2(t) \\
&\quad + 2d(k - 1)\lambda(W(t))\xi(t, 1) + o(1)W(t)) \int_0^1 e^{-x}\xi(t, x) dx \\
&\leq (-\lambda(\overline{\rho}) + o(1))V_2(t) + A(1 + o(1))\xi^2(t, 1) + o(1)W^2(t),
\end{align*}
\] (41)

for some constant $A > 0$.

Finally, we define the Lyapunov function (equivalent to $\|\rho(t, \cdot) - \overline{\rho}\|_{L^2(0, 1)}^2$) as follows:
\[
V(t) := \frac{2A}{\lambda(\overline{\rho})(1 - k^2)} V_1(t) + V_2(t), \quad \forall t \in [0, +\infty).
\] (42)

Again, letting $\varepsilon$ small and noting the continuity of the mapping $\rho_0 \mapsto \rho$, we conclude, from (38), (41) and (42), the proof of Theorem 4.2 in the case $d \geq 1$.

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