REGULARITY THEORY AND ADJOINT-BASED OPTIMALITY CONDITIONS FOR A NONLINEAR TRANSPORT EQUATION WITH NONLOCAL VELOCITY

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Abstract. In this contribution the optimal boundary control problem for a first order nonlinear, nonlocal hyperbolic PDE is studied. Motivated by various applications ranging from re-entrant manufacturing systems to particle synthesis processes, we establish the regularity of solutions for $W^{1,p}$-data. Based on a general $L^2$ tracking type cost functional, the existence, uniqueness, and regularity of the adjoint system in $W^{1,p}$ is derived using the special structure induced from the nonlocal flux function of the state equation. The assumption of $W^{1,p}$- and not $L^p$-regularity comes thereby due to the fact that the adjoint equation asks for more regularity to be well defined. This problem is discussed in detail, and we give a solution by defining a special type of cost functional, such that the corresponding optimality system is well defined.

Key words. conservation law, hyperbolic PDE, nonlocal solution, optimal control theory, adjoint approach, nonlinear transport equation, regularity theory, Lagrange approach

AMS subject classifications. 35L02, 35L04, 35L60, 35L65, 49J20, 49K20

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1. Introduction. The present study is concerned with a first order hyperbolic initial-boundary value problem commonly cited in the context of highly re-entrant manufacturing systems, originally introduced in [AMR+06]. Motivated by problems occurring in the production of semiconductors, Coron, Kawski, and Wang considered in [CKW10] the analytic properties of this class of scalar nonlinear PDEs. Their modeling comprises in particular a nonlocal dependence of the convective term on the current solution. Since this special structure covers various applications and is capable of being transferred to network settings [HKP07, GHKL05], further investigation is indispensable for a comprehensive analysis of this important class of problems.

The present contribution builds upon the work of [CKW10], dealing with the following continuum description of an influx controlled supply chain model:

\begin{equation}
\frac{\partial}{\partial t} \varrho(t,x) + \lambda(W(t,\varrho)) \varrho_x(t,x) = 0, \quad (t,x) \in (0,T) \times (0,1),
\end{equation}

where $\lambda(W(t,\varrho))$ defines a nonlocal velocity function using information on the entire state variable at time $t$, e.g.,

\begin{equation}
W(t,\varrho) := \int_0^1 \varrho(t,s)\,ds, \quad t \in [0,T].
\end{equation}
In the context of manufacturing systems, the boundary data correspond to the influx acting as the controlled variable, complemented by an initial configuration:

\begin{align}
\lambda(W(t, \varrho)) \varrho(t, 0) &= a(t), & t \in (0, T), \\
\varrho(0, x) &= w(x), & x \in (0, 1).
\end{align}

For further reference concerning the modeling of supply chains, we refer the reader to [vdBLR08, AMR+06] and the references therein. In [CHM11] the authors consider a generalized model over space horizon \( \mathbb{R}^N \) for given \( N \in \mathbb{N} \). In Chapter 3 they exactly consider the problem we will also investigate in the following, deduce optimality conditions, and give results for the differentiability of the control to state mapping. However, they consider a different kind of weak solution and prove different results regarding regularity of solutions. Also, due to the assumption of an infinite space horizon, their work is different from the results we obtain.

The considered framework not only suits the described situation of production sites, but also applies to the quite different setting of particle synthesis processes. Here population balance equations represent a state-of-the-art model for the synthesis of polydisperse particulate products. Modeling the phenomena of growth coupled to a mass balance results in a PDE describing the evolution of the particle size distribution in time.

Particles are growing in a supersaturated solution due to homogeneous, transport-dominated growth mechanisms. This results in a decrease of the residual educt concentration strongly depending on the number of currently present particles. Since the growth rate is in turn essentially determined by the current educt concentration, we have again the nonlocal correlation of the convective term to an integral expression evaluating the entire state variable. Closing this remark, we point out that in this case the assumed initial condition corresponds to present seed particles, whereas the boundary data reflect, for example, either the nucleation rate inserting new particles or the flow rate of a feed stream (see, e.g., [Ram00]).

Questions concerning the possibility of precisely controlling the production steps emerge genuinely as a fundamental challenge in industry. The main control elements to influence the process typically consist in varying the influx or the composition of feed streams, respectively. Therefore, the acquisition of optimal control strategies for the considered systems is essential for obtaining a high economic efficiency.

Concerning control problems in the context of particle synthesis, we refer the reader to, for instance, [CELM08, VR06]. For the particular case of production systems, La Marca et al. considered in [LAHR10] the problem of controlling the continuum supply chain model. In their approach a formal adjoint calculus is used for optimally tracking a demand rate as well as a backlog tracking. Their paper ends with a numerical simulation. The problem therein—as we already pointed out in the abstract—that the adjoint PDE, emerging from the formal Lagrange approach, in general asks for more regularity to be well defined. We will discuss this problem in detail remarking that the formal Lagrange approach in [LAHR10] may be numerically justified (due to smoothing effects) but is not applicable without further assumptions on the tracking-type functions and, in particular, on the cost functional. Nevertheless, in section 3 we perform a very similar calculation to obtain the optimality system, but focus on the regularity of the adjoint PDE and its relation to the compatibility conditions at the corners \((0, 0)\) and \((T, 1)\) of the space-time horizon.

Systems closely related to the special structure of the adjoint system are reported in [AD03, Li88, CSu95] treating different aspects against the background of population
dynamics. In [CKW10], the existence of optimal controls to the former mentioned control problem is proven; additionally, a time-optimal control between equilibrium states has been identified. Recently, the results have been extended in [CW12] by Coron and Wang to the case of state and nodal profile controllability. Results for an output feedback stabilization can be found in [CW13].

In the present contribution also the aforementioned optimal boundary control problem is studied using the corresponding Lagrange multiplier. Therefore, we establish rigorously the existence, uniqueness, and regularity properties of the governing state equation as well as of the corresponding adjoint system in a general $W^{1,p}$-setting. The surprising higher regularity assumption will become clear when we provide the adjoint PDE in section 3.

Besides a proof of the well-posedness with respect to the initial data, in particular the introduced Lagrange formalism is justified.

2. Problem formulation. As outlined in the introduction, the described class of applications allows for a uniform modeling. The governing state equation consists in a first order hyperbolic PDE. Pointing out the nonlocal dependence of the convective term on the current solution, the presented framework demands a nonstandard mathematical treatment. Various challenges inherently arise concerning the existence and regularity properties of solutions depending on the choice of spaces.

Definition 2.1 (basic definitions and notation). We define $\mathbb{R}^+ := \{ x \in \mathbb{R} : x > 0 \}$, $\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{ 0 \}$, and $T \in \mathbb{R}^+$. Furthermore, let $\Omega_T := (0, T) \times (0, 1)$ and $q : \Omega_T \to \mathbb{R}$ be a weakly differentiable function on $\Omega_T$. Then we denote by $\dot{q}(t, x)$ the partial derivative with respect to the first variable, the so-called time variable, and with $q_x(t, x)$ the partial derivative with respect to the second, spatial variable.

For a precise statement of the optimal control problem, we introduce the Banach space $H$ which specifies the postulated regularity of the forward and backward problems. Additionally, we define the sets $L^p_+$ and $W^{1,p}_+$.

Definition 2.2 ($H$, $L^p_+$, and $W^{1,p}_+$). Let $p \in [1, \infty)$, $\Omega \subset \mathbb{R}^n$ be open and bounded in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then we define

$$H := C([0, 1]; H^1(0, T)) \cap C([0, T]; H^1(0, 1)),$$

$$L^p_+(\Omega) := \{ y \in L^p(\Omega) : y \geq 0 \ a.e. \}, \quad W^{1,p}_+(\Omega) := \{ y \in W^{1,p}(\Omega) : y \geq 0 \ a.e. \}.$$

In the following, we consider the optimal boundary/initial control problem

$$\min_{w \in W, a \in A} J(w, a)$$

subject to

$$\dot{\varrho}(t, x) = -\lambda(W(t, \varrho))\varrho_x(t, x), \quad (t, x) \in \Omega_T,$$

$$\varrho(0, x) = w(x), \quad x \in (0, 1),$$

$$\varrho(t, 0) = \frac{a(t)}{\lambda(W(t, \varrho))}, \quad t \in (0, T),$$

where $W(t, \varrho)$ is defined by the integral expression

$$W(t, \varrho) := \int_0^1 \varrho(t, s) \, ds, \quad t \in [0, T],$$
and the cost functional is given by

\[
J(w, a) := \frac{\alpha}{2} \|a - a_d\|^2_{L^2(0, T)} + \frac{\delta}{2} \|w - \varrho_0, d\|^2_{L^2(0, 1)} + \frac{\gamma}{2} \|g(T, \cdot) - \varrho_{T, d}\|^2_{L^2(0, 1)}
\]

\[
+ \frac{\lambda}{2} \|\Delta(W(\cdot, 0) + g(\cdot, 1) - \varrho_{T, d})\|^2_{L^2(0, T)},
\]

with \(\lambda \in C^1(\mathbb{R}^+, \mathbb{R}^+), W := L^2_0(0, 1), \) and \(A := L^2_0(0, T)\). The cost functional \(J\) is well defined since by [CKW10, Theorem 2.3 and Remark 2.6] we have for \(\alpha > 0\) that the higher regularity solutions for forward and backward PDEs. Let us furthermore anticipate at this point that the tracked values of \(\varrho\) at \(x = 1\) and \(t = T\), respectively. Additionally, \(a_d\) and \(\varrho_{0, d}\) are enclosed in order to be able to prescribe an average inflow or a desired initial condition.

One may ask why we assume a higher regularity on the tracking-type functions \(\varrho_{T, d}, \varrho_{0, d}, \ldots\). As we shall see in section 3, the adjoint system is only defined for \(H^1\) regular initial and boundary data, and the tracking type functions are part of this boundary data. See also Remark 3.5.

**Remark 2.3 (existence of a minimizer).** By the usual methods we can show (assuming \(\alpha > 0\)) that there is a minimizer of the optimal control problem. First notice that due to the fact that the term \(\frac{\lambda}{2} \|\Delta(W(\cdot, 0) + g(\cdot, 1) - \varrho_{T, d})\|^2_{L^2(0, T)}\) is part of the cost functional, we can easily conclude the boundedness of a minimizing sequence \(a_n\). Thus, there is a weak convergent subsequence, and we can, for instance, furthermore follow the proof of existence of a minimizer in [CKW10, Theorem 3.1].

**Remark 2.4 (the particular choice of the cost functional).** The cost functional (2.6) is intended to be kept as general as possible. However, one may ask why we did not pick as cost functional the usual \(L^2\)-tracking-type functional as done, for instance, in [LAHR10]. Section 3 shows that the special structure in the cost functional, especially the term

\[
\|\lambda(W(\cdot, \varrho) + g(\cdot, 1) - \varrho_{T, d})\|^2_{L^2(0, T)},
\]

leads to a well-posed optimality system, that is, a system where the compatibility conditions at the corner \((T, 1)\) and \((0, 0)\) are satisfied and where we have, thus, \(H^1\) solutions for forward and backward PDEs. Let us furthermore anticipate at this point that the higher regularity \(H^1\) is necessary for the adjoint PDE to be well defined. For that, see section 3 and the computation of the optimality system as well as Remark 3.5.

In order to obtain this well-posedness it is necessary to make the following assumptions. Since the Sobolev embedding theorem \(H^1(\Omega) \hookrightarrow C(\overline{\Omega})\) holds for every real bounded open interval \(I\) (see [Evans10, section 5.6, Theorem 5, p. 269] or [AF03, Theorem 4.12, Parts I and II, p. 85]) and \(a_d, \varrho_{1, d} \in H^1(0, T)\) and \(\varrho_{T, d}, \varrho_{0, d} \in H^1(0, 1)\), we can define a representative of each of these functions which belongs to the space \(C([0, T]), C([0, 1])\), respectively.

**Assumption 2.5 (compatibility conditions on \(a_d, \varrho_{0, d}\) and \(\varrho_{1, d}, \varrho_{T, d}\) and the weights in the cost functional).** For continuity reasons, we require the two compatibility conditions

\[
a_d(0) = \lambda(W(0, \varrho_{0, d})) \varrho_{0, d}(0), \quad \varrho_{1, d}(T) = \varrho_{T, d}(1)
\]
to hold at the origin \((0,0)\) and at the final point \((T,1)\), respectively, and the conditions on the weights in the cost functional, that is,
\[
\alpha = \lambda(W(0, \varrho_0, a))^{-1} \varepsilon, \quad \gamma = \delta.
\]

It will become clear in due course—to be precise in Remark 3.5—that Assumption 2.5 is reasonable in this context.

3. **Formal computation of KKT conditions and the adjoint problem.** In the following we define a Lagrange functional associated with the described optimal control problem using the \(L^2\) inner product to derive first order necessary conditions. Introducing the Lagrange multiplier \(p\) for the PDE constraint, we determine the adjoint solution of the optimality system (2.1)–(2.6). Initially, we assume sufficient regularity on the solution \(\varrho\) as well as for the Lagrange multiplier \(p\) which will be justified later in sections 4 and 5, respectively.

Recalling the Banach space \(H := C([0,T]; H^1(0,1)) \cap C([0,1]; H^1(0,T))\) from Definition 2.2, we define an appropriate Lagrange functional.

**Definition 3.1** (definition of the Lagrange functional). The functional
\[
\mathcal{L} : \left\{ \begin{array}{ll}
H^1(0,1) \times H^1(0,T) \times H \times H \times H^1(0,1) \times H^1(0,T) & \to \mathbb{R},
\end{array} \right.
\]
\[
z := (w, a, \varrho, p_1, p_2) \mapsto \mathcal{L}(w, a, \varrho, p_1, p_2),
\]
with
\[
\mathcal{L}(z) := J(\varrho, w, a) - \langle \varrho, \dot{\varrho} + \lambda(W(\cdot, \varrho))p_2 \rangle_{L^2(\Omega_T)}
\]
\[
+ \langle p(T, \cdot), \varrho(T, \cdot) \rangle_{L^2(0,1)} - \langle p(0, \cdot), \varrho(0, \cdot) \rangle_{L^2(0,1)}
\]
\[
+ \langle p(\cdot, 1), \lambda(W(\cdot, \varrho))\varrho(\cdot, 1) \rangle_{L^2(0,T)} - \langle p(\cdot, 0), \lambda(W(\cdot, \varrho))\varrho(\cdot, 0) \rangle_{L^2(0,T)}
\]
\[
+ \langle \varrho(0, \cdot) - w, p_1 \rangle_{L^2(0,1)} + \langle \lambda(W(\cdot, \varrho))\varrho(\cdot, 0) - a, p_2 \rangle_{L^2(0,T)},
\]

is called the Lagrange functional. Thereby, \(\langle \cdot, \cdot \rangle_{L^2(a,b)}\) represents the \(L^2(a,b)\) inner product.

**Remark 3.2** (Lagrange multipliers). The multipliers \(p, p_1, p_2\) are the Lagrange multipliers for the given constraints. The Lagrange multiplier \(p\) represents the PDE constraint, the multiplier \(p_1\) gives the dependency of the initial control \(w\) to the initial condition \(\varrho(0, \cdot)\), and, finally, \(p_2\) represents the relation between the control \(a\) and the inflow \(\lambda(W(\cdot, \varrho))\varrho(\cdot, 0)\).

Since in the following computation of the KKT conditions (Theorem 3.4) the derivative of \(\lambda(W(t, \varrho))\) with respect to \(\varrho\) is needed, we give the following.

**Lemma 3.3** (Gâteaux derivative of \(\lambda(W(t, \varrho))\) with respect to \(\varrho\)). Let \(\varrho, \tilde{\varrho} \in L^2(\Omega_T)\) be given. Then the Gâteaux derivative of \(\lambda(W(t, \varrho))\) with respect to \(\varrho\) in direction \(\tilde{\varrho}\) is given as
\[
\frac{\partial \lambda(W(\cdot, \varrho))}{\partial \varrho} \tilde{\varrho} = \lambda'(W(\cdot, \varrho)) \int_0^1 \tilde{\varrho}(\cdot, s) \, ds.
\]

In particular, the derivative exists.

**Proof.** By use of the chain rule, reconsidering that \(\lambda\) is \(C^4([0, T]_a, \mathbb{R}^+))\) and \(W(t, \varrho)\) is defined by \(W(t, \varrho) := \int_0^t \varrho(t, s) \, ds\), the proof is complete. \(\square\)

**Theorem 3.4** (KKT conditions). Taking into account Assumption 2.5, the KKT conditions for the optimal control problem (2.1)–(2.6) are given by the following:
• the forward problem,
  
  \[ \dot{\varphi}(t, x) = -\lambda(W(t, \varrho))\varphi_x(t, x), \quad (t, x) \in \Omega_T, \]
  \[ \varphi(0, x) = w(x), \quad x \in (0, 1), \]
  \[ \varphi(t, 0) = \frac{a(t)}{\max(W(t, \varrho))}, \quad t \in (0, T); \]

• the adjoint problem,

  \[ \dot{p}(t, x) = -\lambda(W(t, \varrho))p_x(t, x) - \lambda'(W(t, \varrho)) \left( \int_0^1 p(t, s)p_x(t, s) \, ds \right) \]
  \[ - \lambda'(W(t, \varrho)) \left( \frac{\varphi(t, 1)^2 - \varphi(t, a(t)^2)}{2} \right), \quad (t, x) \in \Omega_T, \]
  \[ p(T, x) = -\delta(\varrho(T, x) - \varrho_{T,a}(x)), \quad x \in (0, 1), \]
  \[ p(t, 1) = -\gamma(\varphi(t, 1) - \varphi(t, a(t))), \quad t \in (0, T); \]

• the coupling conditions,

  \[ p(t, 0) = \alpha(a(t) - a_d(t)), \quad t \in (0, T), \]
  \[ p(0, x) = \varepsilon(w(x) - \varrho_{0,a}(x)), \quad x \in (0, 1). \]

**Proof.** We first compute formally the derivatives of \( \mathcal{L} \) with respect to the functions \( a, w, \) and \( \varrho \) in order to obtain the optimality conditions. We will skip the computation of the derivative with respect to \( p, p_1, \) and \( p_2 \) since they only result in the forward problem. Let \( \tilde{a} \in H^1(0, T) \) be given; then we have

\[
\frac{\partial \mathcal{L}}{\partial a} \tilde{a} = \alpha \langle a(\cdot) - a_d, \tilde{a}(\cdot) \rangle_{L^2(0,T)} - \langle \tilde{a}(\cdot), p_2 \rangle_{L^2(0,T)}
\]

\[
= \langle \tilde{a}, \alpha(a(\cdot) - a_d) - p_2 \rangle_{L^2(0,T)}.\]

Since this term has to be zero for all \( \tilde{a} \in H^1(0, T), \) we can conclude, using the isomorphism of Riesz, that

\[ (3.1) \quad p_2(t) = \alpha(a(t) - a_d(t)) \quad \text{for } t \in (0, T) \text{ a.e.} \]

As we will see in (3.6), we have \( p_2(\cdot) \equiv p(\cdot, 0). \)

We continue with the derivative with respect to \( w \) and obtain, using arguments similar to those above,

\[ p_1(x) = \varepsilon(w(x) - \varrho_{0,a}(x)) \quad \text{for } x \in (0, 1) \text{ a.e.,} \]

and again in (3.5) we will also obtain \( p_1(\cdot) \equiv p(0, \cdot). \) So assuming \( p \) of sufficient regularity, by Assumption 2.5 we obtain the compatibility of \( a \) and \( w \) at the origin \((0, 0).\)

Finally, we determine the Gâteaux derivative of \( \mathcal{L} \) with respect to \( \varrho \) using Lemma 3.3. This computation has to be closely inspected, especially with respect to the variation of \( \varrho \) on the boundary of \( \Omega_T. \) Let \( \tilde{\varrho} \in H \) be given, then we obtain

\[
\frac{\partial \mathcal{L}}{\partial \varrho} \tilde{\varrho} = \langle p(\cdot, 1), \lambda(W(\cdot, \varrho))\tilde{\varrho}(\cdot, 1) \rangle_{L^2(0,T)} + \delta \langle \varrho(T, \cdot) - \varrho_{T,a}, \tilde{\varrho}(T, \cdot) \rangle_{L^2(0,1)}
\]

\[ + \gamma \langle \lambda(W(\cdot, \varrho)) (\varphi(\cdot, 1) - \varphi(T,a)), \tilde{\varrho}(\cdot, 1) \rangle_{L^2(0,T)} + \langle p(T, \cdot), \tilde{\varrho}(T, \cdot) \rangle_{L^2(0,1)}
\]

\[ - \langle p(0, \cdot), \tilde{\varrho}(0, \cdot) \rangle_{L^2(0,1)} + \langle \varrho(0, \cdot), p_1(\cdot) \rangle_{L^2(0,1)} \]
The adjoint problem. To separate the corresponding items, we can choose \( \tilde{\rho} \) problem, while the remaining terms will play their part in the boundary condition of (3.2)

\[
\begin{align*}
+ \langle \lambda(W(\cdot, \rho))\tilde{\rho}(\cdot, 0), p_2 \rangle_{L^2(0,T)} - \langle \lambda(W(\cdot, \rho))\tilde{\rho}(\cdot, 0), p(\cdot, 0) \rangle_{L^2(0,T)} \\
- \langle \tilde{\rho}, \rho + \lambda(W(\cdot, \rho))p_x \rangle_{L^2(\Omega_T)} \\
- \left\langle \lambda'(W(\cdot, \rho)) \int_0^1 \tilde{\rho}(\cdot, s) ds \cdot p_x, \rho \right\rangle_{L^2(\Omega_T)} \\
+ \left\langle \lambda'(W(\cdot, \rho)) \int_0^1 \tilde{\rho}(\cdot, s) ds \cdot \rho(\cdot, 0), p_2 \right\rangle_{L^2(0,T)} \\
- \left\langle \lambda'(W(\cdot, \rho)) \int_0^1 \tilde{\rho}(\cdot, s) ds \cdot \rho(\cdot, 0), p(\cdot, 0) \right\rangle_{L^2(0,T)} \\
+ \left\langle \lambda'(W(\cdot, \rho))\tilde{\rho}(\cdot, 1) \int_0^1 \tilde{\rho}(\cdot, s) ds \cdot \rho(\cdot, 1), 1 \right\rangle_{L^2(0,T)} \\
+ \frac{\gamma}{2} \left\langle \lambda'(W(\cdot, \rho)) \int_0^1 \tilde{\rho}(t, s) ds \cdot (\rho(\cdot, 1) - \rho_{1,d})^2, 1 \right\rangle_{L^2(0,T)} .
\end{align*}
\]

The terms which contain the integral \( \int_0^1 \tilde{\rho}(\cdot, s) ds \) will thereby contribute to the adjoint problem, while the remaining terms will play their part in the boundary condition of the adjoint problem. To separate the corresponding items, we can choose \( \tilde{\rho} \in C^\infty_0(\Omega_T) \) since the necessary optimality conditions require the computed derivative to be zero for all \( \tilde{\rho} \in H \).

As \( C^\infty_0(\Omega_T) \) is a dense subspace of \( C([0,1]; H^1_0(0,T)) \cap C([0,T]; H^1_0(1,1)) \subset H \), and since the boundary and initial/end values are zero, we conclude that

\[
0 = \left\langle \tilde{\rho}, -\rho - \lambda(W(\cdot, \rho))p_x - \lambda'(W(\cdot, \rho)) \int_0^1 \rho(\cdot, s) \cdot p_x(\cdot, s) ds \right\rangle_{L^2(\Omega_T)} \\
+ \left\langle \tilde{\rho}, \lambda'(W(\cdot, \rho))\rho(\cdot, 0)(p_2(\cdot) - p(\cdot, 0)) \right\rangle_{L^2(\Omega_T)} \\
+ \left\langle \tilde{\rho}, \lambda'(W(\cdot, \rho))\rho(\cdot, 1)p(\cdot, 1) \right\rangle_{L^2(\Omega_T)} \\
+ \frac{\gamma}{2} \left\langle \tilde{\rho}, \lambda'(W(\cdot, \rho))\rho(\cdot, 1) - \rho_{1,d} \right\rangle_{L^2(\Omega_T)} \\
\forall \tilde{\rho} \in H
\]

using Fubini's theorem. In particular, we want to point out that due to the nonlocal dependence of the flux function on the entire state variable, we obtain boundary terms in the derivation of the adjoint equation. These terms are a special feature of the given system and may not be neglected in order to get the correct adjoint equation. Sorting also the boundary terms gives

\[
0 = \left\langle \tilde{\rho}(\cdot, 1), \lambda(W(\cdot, \rho))(p(\cdot, 1) + \gamma(\rho(\cdot, 1) - \rho_{1,d})) \right\rangle_{L^2(0,T)} \\
+ \left\langle \tilde{\rho}(T, \cdot), p(T, \cdot) + \delta(\rho(T, \cdot) - \rho_{T,d}) \right\rangle_{L^2(0,1)} + \left\langle \tilde{\rho}(0, \cdot), p(0, \cdot) \right\rangle_{L^2(0,1)} \\
+ \left\langle \tilde{\rho}(\cdot, 0), \lambda(W(\cdot, \rho))(p_2 - p(\cdot, 0)) \right\rangle_{L^2(0,T)} \\
\forall \tilde{\rho} \in H .
\]

At this point the special choice of the cost functional (2.6), as described in Remark 2.4, becomes indispensable. It allows for a decoupling of the tracked value \( \rho(\cdot, 1) - \rho_{1,d}(t) \) at \( x = 1 \) from the nonlocal velocity function \( \lambda(W(\cdot, \rho)) \). Since \( \lambda(W(t, \rho)) \neq 0 \), we set

\[
p(t, 1) = -\gamma(\rho(\cdot, 1) - \rho_{1,d}(t)) \quad \text{for} \quad t \in (0,T) \text{ a.e.} ,
\]

\[
p(T, x) = -\delta(\rho(T, x) - \rho_{T,d}(x)) \quad \text{for} \quad x \in (0,1) \text{ a.e.}
\]
Reconsidering Assumption 2.5, these equations can even be extended pointwise to the closed intervals. Therefore, if \( \varrho_{1,d}(T) = \varrho_{T,d}(1) \) and \( \gamma = \delta \), we obtain
\[
(3.3) \quad p(t, 1) = -\gamma \left( \varrho(\cdot, 1) - \varrho_{1,d}(t) \right) \quad \text{for } t \in [0, T],
(3.4) \quad p(T, x) = -\delta \left( \varrho(T, x) - \varrho_{T,d}(x) \right) \quad \text{for } x \in [0, 1],
\]
since in this case we have \( p(t, 1)|_{t=T} = p(T, x)|_{x=1} \), the compatibility condition of the adjoint equation in \((T, 1)\).

In a next step we consider again Assumption 2.5. By an argument analogous to that above, we find that
\[
(3.5) \quad p_1(x) = p(0, x) \quad \text{for } x \in [0, 1],
(3.6) \quad p_2(t) = p(t, 0) \quad \text{for } t \in [0, T]
\]
hold using the higher regularity of the representations of \( p_1, p_2, \) and \( p \). Conclusively, we can even write
\[
p_2(t) = \alpha(a(t) - a_d(t)) \quad \text{for } t \in [0, T]
\]
using (3.1). Let us now turn our focus to the adjoint problem. We reformulate (3.2) by sorting the terms
\[
\begin{align*}
\left\{ \tilde{\varrho}, \quad & -\dot{\varrho} - \lambda(W(\cdot, \varrho)) p_x - \lambda'(W(\cdot, \varrho)) \int_0^1 \varrho(t, s) p_x(t, s) \, ds \\
& + \lambda'(W(\cdot, \varrho)) \left( \varrho(\cdot, 0) \left( p_2(\cdot) - p(\cdot, 0) \right) + \varrho(\cdot, 1) p(\cdot, 1) \right) \\
& + \lambda'(W(\cdot, \varrho)) \gamma(\varrho(\cdot, 1) - \varrho_{1,d})^2 \right\}_{L^2(\Omega_T)} = 0.
\end{align*}
\]
This equality has to be true for all \( \tilde{\varrho} \in H \). Under the given Assumption 2.5 and using the equalities (3.3)–(3.6) deduced above, we obtain the adjoint PDE as
\[
\begin{align*}
\dot{\varrho}(t, x) &= -\lambda(W(t, \varrho)) p_x(t, x) - \lambda'(W(t, \varrho)) \left( \int_0^1 \varrho(t, s) p_x(t, s) \, ds \right) \\
& - \lambda'(W(t, \varrho)) \left( \tilde{\gamma}(\varrho(t, 1)^2 - \varrho_{1,d}(t)^2) \right), \quad (t, x) \in \Omega_T.
\end{align*}
\]
This concludes the proof of the stated theorem. \( \square \)

Having computed the necessary optimality conditions formally, we are obliged to justify that the regularity of the forward and backward problems are in fact at least \( H \). Since the regularity of the considered systems may also be of interest in different settings, we establish a more general result assuming \( W^{1, p} \)-data for \( p \in [1, \infty] \) and not only \( H^1 \).

Before we give a detailed regularity study of forward and backward PDEs we should mention why we need the higher regularity.

Remark 3.5 (the claimed higher regularity for the adjoint equation). Consider the adjoint PDE
\[
\dot{\varrho}(t, x) = -\lambda(W(t, \varrho)) p_x(t, x) - \lambda'(W(t, \varrho)) \left( \int_0^1 \varrho(t, s) p_x(t, s) \, ds \right) \\
- \lambda'(W(t, \varrho)) \left( \tilde{\gamma}(\varrho(t, 1)^2 - \varrho_{1,d}(t)^2) \right), \quad (t, x) \in \Omega_T,
\]
complemented by the boundary and coupling conditions

\begin{align*}
p(T, x) &= -\gamma(g(T, x) - g_{T,d}(x)), & x \in (0, 1), \\
p(t, 1) &= -\gamma(g(t, 1) - g_{1,d}(t)), & t \in (0, T), \\
p(t, 0) &= a(a(t) - a_d(t)), & t \in (0, T), \\
p(0, x) &= \alpha\lambda(W(0, g_{0,d}))(w(x) - g_{0,d}(x)), & x \in (0, 1),
\end{align*}

and the term on the right-hand side \(\lambda'(W(t, \varrho)) \int_0^1 g(t, s)p_x(t, s) \, ds\). Assuming only \(L^2\)-regularity for the forward PDE will thus ask for \(H^1\)-regularity of the adjoint solution \(p\), since—as is usually done in the weak formulation—the derivative of \(p\) with respect to \(x\) cannot be shifted to a given sufficient regular test function. But if the forward system has only \(L^p\)-regularity, by the coupling also the boundary condition and end data on \(p\) will only have \(L^p\)-regularity, and we cannot expect a higher regular \(p\) on \(\Omega_T\). So we have to choose higher regular spaces, such that this expression is well defined. The reasonable choice thereby is \(H^1\) such that forward and backward PDEs have the postulated regularity. This, on the other hand, leads to compatibility conditions at the point \((0, 0)\) for the forward equation and \((T, 1)\) for the backward equation. Assumption 2.5 guarantees these conditions.

One may ask why we do not perform an integration by parts to shift the derivative on \(\varrho\) and to avoid a higher regularity on \(p\). But then we need higher regularity on \(\varrho\), which can only be given if also \(p\) admits higher regularity, since the initial and boundary values of \(\varrho\) are coupled with \(p\).

Remark 3.6 (velocity function \(\lambda \equiv \text{const}\)). If \(\lambda\) is a constant positive function, the terms multiplied by \(\lambda'\) vanish and we obtain the expected KKT conditions for a linear boundary (and initial) control system. Furthermore, forward and backward problems are of the same structure as would be expected for a linear problem.

4. Regularity of the solution to the forward problem. Starting with the regularity properties of the forward problem, we consider the state equation constraining the optimal control problem (2.1)–(2.4). Since there are, to the best of our knowledge, no appropriate regularity results on the considered problem for \(W^{1,p}\)-data, we extend the statements established in [CKW10] to match the requirements. For \(L^p\)-data as initial and boundary values of the forward problem we obtain the following.

Theorem 4.1 (regularity of the forward problem for \(L^p\)-data). Let \(g_0 \in L^p_k(0, 1)\) and \(a \in L^p_k(0, T)\); then there exists a unique weak solution

\[ \varrho \in C^0([0, T]; L^p(0, 1)) \cap C^0([0, 1]; L^p(0, T)) \]

of the forward problem

\begin{align*}
\varrho(t, x) &= -\lambda(W(t, \varrho))g_x(t, x), & (t, x) \in \Omega_T, \\
\varrho(t, 0) &= \frac{a(t)}{\lambda(W(t, \varrho))}, & t \in (0, T), \\
\varrho(0, x) &= g_{0,x}(x), & x \in (0, 1), \\
W(t, \varrho) &= \int_0^1 \varrho(t, s) \, ds, & t \in [0, T],
\end{align*}

with \(\lambda \in C^1(\mathbb{R}_0^+, \mathbb{R}^+)\), which is nonnegative almost everywhere in \(\Omega_T\). Thereby, a weak solution is a solution in usual form as defined in [CKW10, Definition 2.1].
Proof. See [CKW10, Theorem 2.3 and Remark 2.6].

Our strategy of proving the regularity results in Theorem 4.7 for the stronger $W^{1,p}$-setting is now to use the solution given in [CKW10] for $L^p$-data. The solution therein is constructed by characteristic curves and the propagation of initial and boundary values with the help of these characteristics. This solution is explicitly given and can be estimated appropriately to establish the claimed regularity.

In order to clarify the notation, we now define these characteristics, which are used in the proofs of the following lemmas and theorems, and summarize their related properties.

**Definition 4.2 (characteristics).** Let $\alpha \in C^0([0,T]; \mathbb{R}^+)$. First of all, we define the characteristic $\xi: [0,T] \rightarrow \mathbb{R}^+$ emanating from the origin by

$$\dot{\xi}(t) = \alpha(t), \quad \xi(0) = 0$$

and, with the help of $\xi$, the characteristic $\xi(t,x) : [0,T] \rightarrow \mathbb{R}^+$ ending in $(t,x) \in \Omega_T$:

$$\dot{\xi}(t,x)(t) = \alpha(t), \quad \xi(t,x)(t) = x.$$  

Furthermore, we define the functions $\eta, \tau$ mapping the spatial variable $x \in [0,1]$ for a given time $t \in [0,T]$ to the corresponding initial or boundary value (see Figure 1):

$$\eta: \begin{cases} \{ (t,x) \in \Omega_T : x \geq \xi(t), \ t \leq \xi^{-1}(1) \} & \rightarrow [0,1], \\ (t,x) & \mapsto \xi_{(t,x)}(0) \end{cases}$$

and

$$\tau: \begin{cases} \{ (t,x) \in \Omega_T : x < \xi(t) \} & \rightarrow [0,T], \\ (t,x) & \mapsto \xi^{-1}_{(t,x)}(0). \end{cases}$$

![Fig. 1. Definition of characteristics $\xi(t)$ and $\xi(t,x)(t)$, $(t_0,x_0), (t_1,x_1) \in \Omega_T$.](image)

**Remark 4.3 (related properties of the characteristics).** We present selected properties of the defined characteristics related to our context:

(i) Since $t \mapsto \xi(t)$ and also $t \mapsto \xi_{(t,x)}(t)$ are functions in $C^1([0,T], \mathbb{R}^+)$ with strictly positive derivative $\alpha(t)$, the characteristic $\xi(t)$ is invertible and justifies the definition in (4.4).
(ii) We obtain an explicit formulation by integration:
\[ \xi(t) = \int_0^t \alpha(s) \, ds \quad \text{and} \quad \xi(t,x)(t) = x + \int_t^1 \alpha(s) \, ds. \]

(iii) We have \( \xi(t,x)(t) = \xi(t) + \int_t^1 \alpha(s) \, ds = \xi(t) + \xi(t) - \xi(t) = \xi(t), \) so the definition in (4.2) generalizes the definition in (4.1).

(iv) In order to express \( \eta \) in (4.3) solely based on \( \xi(t) \), we can write
\[ \eta(t,x) = x - \xi(t) \quad \text{for} \quad (t, x) \in \{(t, x) \in \Omega_T : x \geq \xi(t), \ t \leq \xi^{-1}(1)\}. \]

(v) To express \( \tau \) in (4.4) accordingly using only \( \xi(t) \), we perform for \( x < \xi(t) \)
the transformation \( \tau(t,x) = \xi_{(t,x)}^{-1}(0) \implies \xi_{(t,x)}(\tau(t,x)) = 0. \) Now,
\[ \xi_{(t,x)}(\tau(t,x)) = x + \int_t^{\tau(t,x)} \alpha(s) \, ds = x + \int_0^{\tau(t,x)} \alpha(s) \, ds - \int_0^t \alpha(s) \, ds = x + \xi(\tau(t,x)) - \xi(t) = 0 \]
and, therefore, \( \xi(\tau(t,x)) = \xi(t) - x \implies \tau(t,x) = \xi^{-1}(\xi(t) - x), \) so we can finally write
\[ \tau(t,x) = \xi^{-1}(\xi(t) - x) \quad \text{for} \quad (t, x) \in \{(t, x) \in \Omega_T : x < \xi(t)\}. \]

(vi) For a given time \( t \in [0, T] \) the functions \( \tau \) and \( \eta \) define a mapping of the spatial variable \( x \in [0, 1] \) to the corresponding initial or boundary value later used for particular integral transformations:
\[
\begin{align*}
\tau(t, \cdot) : & \quad [0, \xi(t)) \to [0, t] \\
\eta(t, \cdot) : & \quad [\xi(t), 1] \to [0, 1 - \xi(t)]
\end{align*}
\]
for \( 0 \leq t \leq \xi^{-1}(1). \)

Moreover, as shown in Figure 2, we have iteratively for \( t > \xi^{-1}(1) \) and for \( n \in \mathbb{N} \)
\[ \tau(t, \cdot) : \quad [0, 1] \to (\xi^{-1}(\xi(t) - n), t] \quad \text{for} \quad \xi^{-1}(n) < t \leq \xi^{-1}(n + 1). \]

\[ \text{Fig. 2. Iterative definition of characteristics for } t > \xi^{-1}(1). \]
(vii) For the partial derivatives of \( \eta(t, x) \) we obtain

\[
\frac{\partial \eta(t, x)}{\partial x} = 1 \quad \text{and} \quad \frac{\partial \eta(t, x)}{\partial t} = -\xi'(t) = -\alpha(t).
\]

(viii) For the partial derivatives of \( \tau(t, x) \) we use the following equality, taking first the derivative with respect to time \( \xi(\tau(t, x)) = \xi(t) - x \implies \xi'(\tau(t, x)) \frac{\partial \tau(t, x)}{\partial t} = \xi'(t) = \alpha(t) \) in order to get

\[
\frac{\partial \tau(t, x)}{\partial t} = \frac{\alpha(t)}{\alpha(\tau(t, x))}.
\]

Furthermore, \( \xi(\tau(t, x)) = \xi(t) - x \implies \xi'(\tau(t, x)) \frac{\partial \tau(t, x)}{\partial x} = -1 \), yielding

\[
\frac{\partial \tau(t, x)}{\partial x} = -\frac{1}{\alpha(\tau(t, x))}.
\]

In the case under consideration, we need a stronger result to justify the derivation of the requested optimality conditions; assuming \( a \in W^{1,p}_+(0, T) \) and \( \varrho_0 \in W^{1,p}_+(0, 1) \), we propose the regularity

\[
\varrho \in C^0([0, T]; W^{1,p}_+(0, 1)) \cap C^0([0, 1]; W^{1,p}_+(0, T))
\]

for the solution of the forward problem. Since we will show in the main result of this section, Theorem 4.7, that \( \varrho_x \) and \( \varrho \) belong to \( C^0([0, T]; L^p(0, 1)) \cap C^0([0, 1]; L^p(0, T)) \), we will make use of the following lemma to conclude this proof.

**Lemma 4.4.** Let \( p \in [1, \infty) \) be given, let \( \varrho : \Omega_T \to \mathbb{R} \) be a function with \( \varrho(0, \cdot) \in W^{1,p}_+(0, 1) \), \( \varrho(\cdot, 0) \in W^{1,p}_+(0, T) \), and let \( \varrho_x, \varrho \in C^0([0, T]; L^p(0, 1)) \cap C^0([0, 1]; L^p(0, T)) \). Then we have

\[
\varrho \in C^0([0, T]; W^{1,p}_+(0, 1)) \cap C^0([0, 1]; W^{1,p}_+(0, T)).
\]

**Proof.** We perform the following estimation for \( t, s \in [0, T] \):

\[
\| \varrho(t, \cdot) - \varrho(s, \cdot) \|_{W^{1,p}_+(0, 1)} \leq \int_0^1 |\varrho(t, x) - \varrho(s, x)|^p \, dx + \| \varrho_x(t, x) - \varrho_x(s, x) \|^p \, dx.
\]

The second term in the integral vanishes for \( t \to t \) due to the fact that we assumed \( \varrho_x \in C^0([0, T]; L^p(0, 1)) \cap C^0([0, 1]; L^p(0, T)) \), so we concentrate only on the first term on the right-hand side and do some further calculations to obtain an estimation depending on the \( L^p \)-norm of \( \varrho_x \). Using the fundamental theorem of calculus, we have

\[
\int_0^1 |\varrho(t, x) - \varrho(s, x)|^p \, dx = \int_0^1 |\varrho(t, 0) - \varrho(s, 0) + \int_0^s \varrho_x(t, s) - \varrho_x(t, 0) \, ds|^p \, dx \\
\leq \int_0^1 \left( 2 \max \left\{ |\varrho(t, 0) - \varrho(s, 0)|, \int_0^s |\varrho_x(t, s) - \varrho_x(t, 0)| \, ds \right\} \right)^p \, dx \\
\leq 2^p \int_0^1 \left( \max \left\{ |\varrho(t, 0) - \varrho(s, 0)|, \int_0^1 |\varrho_x(t, s) - \varrho_x(t, 0)| \, ds \right\} \right)^p \, dx.
\]

The right-hand side in this inequality also vanishes due to the assumptions on \( \varrho(\cdot, 0) \in W^{1,p}_+(0, T) \to C^0([0, T]) \) (see again [Eva10, section 5.6, Theorem 5, p. 269] or also [AF03, Theorem 4.12, Parts I and II, p. 85]) and \( \varrho_x \in C^0([0, T]; L^p(0, 1)) \cap C^0([0, 1]; W^{1,p}_+(0, T)) \).
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To show that \( \varrho \in C^0([0, 1]; W^{1,p}(0, T)) \) results in an analogous estimation exchanging \( x \) and \( t \).

**Remark 4.5 (compatibility condition).** The assumptions of this lemma, namely, \( \varrho(0, \cdot) \in W^{1,p}(0, 1) \) for the initial data and \( \varrho(\cdot, 0) \in W^{1,p}(0, T) \) for the boundary data, include the continuity at the origin. Since we have for every open interval \( I \subset \mathbb{R} \) a continuous embedding of \( W^{1,p}(I) \) in \( C^0(I) \) (see, e.g., [Eva10, section 5.6, Theorem 5, p. 269] or [AF03, Theorem 4.12, Parts I and II, p. 85]), the continuity at the origin is inherently contained.

Before proving the claimed properties of the forward equation, we consider the regularity of the following linear transport equation. Although a specification of \( L^p \)-data is uncommon in the general theory, we encounter the following setting in the proof of our theorem. Therefore, anticipating this subproblem, the result is stated below.

**Lemma 4.6 (regularity of a linear transport equation).** We consider the linear transport equation

\[
\dot{q}(t, x) + v(t)q_x(t, x) = 0, \quad (t, x) \in (0, T) \times (0, 1),
\]

\[
q(0, x) = q_0(x), \quad x \in (0, 1),
\]

\[
q(t, 0) = a(t), \quad t \in (0, T),
\]

where \( v \in C^0([0, T], \mathbb{R}^+) \), \( a \in L^p(0, T) \), and \( q_0 \in L^p(0, 1) \). Then the following regularity holds:

\[
q \in C^0([0, T]; L^p(0, 1)) \cap C^0([0, 1]; L^p(0, T)).
\]

**Proof.** Using the characteristics specified in Definition 4.2 with \( v \equiv \alpha \), for \( t \leq \xi^{-1}(1) \) we can state the weak solution of the linear transport equation as

\[
q(t, x) = \begin{cases} a(\tau(t, x)) & \text{for } x \leq \xi(t), \\ q_0(q(t, x)) & \text{for } x > \xi(t). \end{cases}
\]

Now, one can easily estimate the regularity and show that \( q \) is indeed a weak solution in the usual sense. We omit these details here.

Finally, we obtain the main theorem of this section.

**Theorem 4.7 (regularity of the forward problem for \( W^{1,p} \)-data).** Let \( p \in [1, \infty) \), \( \varrho_0 \in W^{1,p}_+(0, 1) \), \( a \in W^{1,p}_+(0, T) \), and let the compatibility condition \( \varrho_0(0) = a(0) \) hold. Then there exists a weak solution

\[
\varrho \in C^0([0, T]; W^{1,p}(0, 1)) \cap C^0([0, 1]; W^{1,p}(0, T))
\]

satisfying

\[
\dot{\varrho}(t, x) = -\lambda(W(t, \varrho))\varrho_x(t, x), \quad (t, x) \in \Omega_T,
\]

\[
\varrho(t, 0) = \frac{a(t)}{\lambda(W(t, \varrho))}, \quad t \in (0, T),
\]

\[
\varrho(0, x) = \varrho_0(x), \quad x \in (0, 1),
\]

where

\[
W(t, \varrho) = \int_0^1 \varrho(t, s) \, ds, \quad t \in [0, T],
\]
with \( \lambda \in C^1(\mathbb{R}_+^+, \mathbb{R}^+) \). Furthermore, the solution is unique and nonnegative.

Proof. Assuming sufficient regularity, we differentiate the forward PDE with respect to \( x \). We obtain the following initial boundary value problem:

\[
\begin{align*}
\dot{w}(t, x) &= -\lambda(W(t, \varrho))w_x(t, x), \quad (t, x) \in \Omega_T, \\
\dot{w}(t, 0) &= -\lambda(\varrho(t, 0))w_x(t, 0), \quad t \in (0, T), \\
w_x(0, x) &= \varrho_t(0, x), \quad x \in (0, 1), \\
W(t, \varrho) &= \int_0^1 \varrho(t, s) ds, \quad t \in [0, T].
\end{align*}
\]

From [CKW10, p. 1349] we know that for \( L^p \) initial and boundary data, \( t \mapsto W(t, \varrho) \) is an absolutely continuous function with derivative

\[
W'(t, \varrho) = \begin{cases} 
\frac{a(t)}{\lambda(W(t, \varrho))} - \lambda(W(t, \varrho))a(t), & 0 \leq t \leq \xi^{-1}(1), \\
\frac{a(t)}{\lambda(W(t, \varrho))} - \frac{\xi(t)\lambda(n(\xi(t)-n))}{\lambda(n(\xi(t)-n))}, & \xi^{-1}(n) < t \leq \xi^{-1}(n+1), \quad n \in \mathbb{N},
\end{cases}
\]

where \( \xi(t) \) is again the characteristic from Definition 4.2. Due to our assumption of \( W^{1,p} \)-data, the stated result applies in our case as well.

Since \( \lambda \in C^1(\mathbb{R}_+^+, \mathbb{R}^+) \) and \( W \in C^0([0, T]) \), the function \( t \mapsto \lambda(W(t, \varrho)) \) is continuous on \([0, T] \). Furthermore, \( \varrho_0 \in L^p(0, 1) \) and

\[
-\left( \frac{a(t)}{\lambda(W(t, \varrho))} \right)' = \frac{\lambda'(W(t, \varrho))W'(t)a(t) - \lambda(W(t, \varrho))a'(t)}{\lambda(W(t, \varrho))^2}.
\]

Now we show that this function is bounded in \( L^p \):

\[
\left\| \frac{\lambda'(W(t, \varrho))W'(t)a(t) - \lambda(W(t, \varrho))a'(t)}{\lambda(W(t, \varrho))^2} \right\|_{L^p(0, T)}^p \leq \left\| \frac{\lambda'(W(t, \varrho))W'(t)a(t)}{\lambda(W(t, \varrho))^2} \right\|_{L^p(0, T)}^p + \left\| \frac{a'(t)}{\lambda(W(t, \varrho))} \right\|_{L^p(0, T)}^p.
\]

Let \( I \subset \mathbb{R} \) be an open interval. Then we have the continuous embedding of \( W^{1,p}(I) \) in \( C^0(I) \) (see Remark 4.5) and can therefore estimate

\[
\left\| \frac{\lambda'(W(t, \varrho))W'(t)a(t)}{\lambda(W(t, \varrho))^2} \right\|_{L^p(0, T)}^p \leq \frac{1}{\lambda_{\min}} \left\| \lambda'(W(\cdot, \varrho)) \right\|_{L^p(0, T)}^p \left\| a \right\|_{L^p(0, T)}^p \left\| W' \right\|_{L^p(0, T)}^p + \frac{1}{\lambda_{\min}} \left\| a' \right\|_{L^p(0, T)}^p < \infty.
\]

In the last inequality we also estimated \( \lambda(W(\cdot, \varrho)) \) in the uniform norm. This is possible since \( W(t, \varrho) = \int_0^1 \varrho(t, s) ds \) is an absolute continuous function (see above) and takes its maximum and minimum on \([0, T]\). Therefore, \( \lambda(W(t, \varrho)) \) also attains its (maximum) minimum on \([0, T]\), which we denote by \( \lambda_{\min} \).

That \( W'(\cdot, \varrho) \) is bounded with respect to the \( L^p \)-norm can be shown as follows:

From the definition of the characteristics we find that \( \xi(t) = \int_0^t \lambda(W(s, \varrho)) ds \). Since the function \( t \mapsto \lambda(W(t, \varrho)) \) is continuous, we have \( \xi \in C^1([0, T]) \). Furthermore \( \xi'(t) = \lambda(W(t, \varrho)) \) is positive, \( \xi^{-1} \) exists, and hence, by the implicit function theorem,
the function \( t \mapsto \xi^{-1}(\cdot) \) is also continuously differentiable. So we can perform the following estimation, again using \( W^{1,p}(0,T) \hookrightarrow C^0[0,T] \):

\[
\| W' \|^p_{L^p(0,T)} = \int_0^{\xi^{-1}(1)} |a(t) - \xi'(t)\varrho_0(1 - \xi(t))|^p \, dt + \int_{\xi^{-1}(1)}^T |a(t) - \frac{\xi'(t)\varrho_0(\xi^{-1}(\xi(t)-1))}{\xi'(\xi^{-1}(\xi(t)-1))}|^p \, dt \\
\leq \| \varrho(\cdot) - \xi'(\cdot)\varrho_0(1 - \xi(\cdot)) \|^p_{L^\infty(0,T)} \cdot \int_0^{\xi^{-1}(1)} \, dt \\
+ \left\| a(\cdot) - \frac{\xi'(\cdot)\varrho_0(\xi^{-1}(\xi(t)-1))}{\xi'(\xi^{-1}(\xi(t)-1))} \right\|_{L^\infty(0,T)} \cdot \int_{\xi^{-1}(1)}^T \, dt \\
\leq T \| a(\cdot) - \xi'(\cdot)\varrho_0(1 - \xi(\cdot)) \|^p_{L^\infty(0,T)} + T \left\| a(\cdot) - \frac{\xi'(\cdot)\varrho_0(\xi^{-1}(\xi(t)-1))}{\xi'(\xi^{-1}(\xi(t)-1))} \right\|_{L^\infty(0,T)}
\]

The right-hand side is bounded (thanks to the embedding theorems), and thus \( W' \in L^p(0,T) \).

The transformed PDE is a transport equation with \( L^p \) initial and boundary data and a velocity function \( \lambda(W(\cdot, \varrho)) \in C^0[0,T] \). This equation—using the results presented in Lemma 4.6—admits a unique solution \( \varrho_\varepsilon \in C^0([0,T]; L^p(0,1)) \cap C^0([0,1]; L^p(0,T)) \). Using the continuity of \( \lambda(W(\cdot, \varrho)) \) in the PDE, we obtain the same regularity result for \( \varrho_\varepsilon \). So we can finally conclude with the help of Lemma 4.4 that

\[
\varrho \in C^0([0,T]; W^{1,p}(0,1)) \cap C^0([0,1]; W^{1,p}(0,T)).
\]

The fact that the solution is nonnegative almost everywhere as claimed in the theorem can be deduced with the help of Theorem 4.1, since \( W_+^{1,p} \subset L^p_+ \).

\textbf{Remark 4.8 (nonnegativity of the solution \( \varrho \)).} Since \( \varrho \in C^0([0,T]; W^{1,p}(0,1)) \hookrightarrow C^0([0,T]; C^0([0,1])) \simeq C^0_0(\Omega_T) \), we can even find a representation of \( \varrho \) which is continuous on \( \Omega_T \). Therefore, the nonnegativity for the representation holds everywhere in \( \Omega_T \).

\section{5. Regularity of the solution to the adjoint problem}

Next we consider the solution to the adjoint problem defined in Theorem 3.4 again in a more general setting. We analyze the following system for given \( W^{1,p} \) boundary and final data:

\[
\begin{aligned}
\dot{p}(t,x) &= -\lambda(W(t,\varrho))p_x(t,x) - \lambda'(W(t,\varrho)) \int_0^1 \varrho(t,s)p_x(t,s) \, ds \\
&\quad - \lambda'(W(t,\varrho)) \left( \frac{1}{2}(\varrho(t,1)^2 - \varrho_1(t)^2) \right), \\
p(T,x) &= p_0(x), \\
p(t,1) &= u(t), \\
p(t,1) &= u(t), \\
p(T,1) &= u(T).
\end{aligned}
\]

For a more convenient representation, we transform the backward problem using the following affine mappings:

\[
\mathcal{T} : [0,T] \to [0,T], \ t \mapsto T - t \quad \text{and} \quad \mathcal{X} : [0,1] \to [0,1], \ x \mapsto 1 - x.
\]

\textbf{Definition 5.1 (transformed PDE).} For \( \mathcal{T}, \mathcal{X}, \) and \( p \) as defined before we set

\[
\hat{p}(t,x) := p(T(t),\mathcal{X}(x)) \quad \forall (t,x) \in \Omega_T.
\]

Then we obtain the following.
Lemma 5.2. The function $\tilde{p}$, defined on $\Omega_T$, fulfills the following initial boundary value problem:

\begin{align}
\tilde{p}(t,x) + \alpha(t)\tilde{p}_x(t,x) &= -\int_0^1 \gamma(t,s)\tilde{p}_x(t,s)\,ds - \varphi(t), \quad (t,x) \in \Omega_T, \\
\tilde{p}(0,x) &= p_0(\lambda(x)), \quad x \in (0,1), \\
\tilde{p}(t,0) &= u(T(t)), \quad t \in (0,T),
\end{align}

where

$$
\alpha(t) := \lambda(W(T(t),\varrho)), \quad \gamma(t,x) := \lambda'(W(T(t),\varrho))\varrho(T(t),\lambda(x)), \\
\varphi(t) := \lambda'(W(T(t),\varrho))\frac{1}{2}(\varrho(T(t),1)^2 - \varrho_1(T(t))^2)
$$

are independent with respect to $\tilde{p}$.

Proof. The proof can be done by simple calculations.

Since (5.1)–(5.3) is equivalent to the adjoint problem (the transformations in time and space are affine and bijective) the obtained regularity results apply accordingly for both systems.

Moreover, reconsidering Remark 4.5, we require the stated compatibility condition in the following theorem as well.

Theorem 5.3 (existence and regularity for the solution to the adjoint problem). Let us consider

\begin{align}
\tilde{p}(t,x) + \alpha(t)\tilde{p}_x(t,x) &= -\int_0^1 \gamma(t,s)\tilde{p}_x(t,s)\,ds - \varphi(t), \quad (t,x) \in \Omega_T, \\
\tilde{p}(0,x) &= \tilde{p}_0(x), \quad x \in (0,1), \\
\tilde{p}(t,0) &= u(t), \quad t \in (0,T).
\end{align}

If $\tilde{p}_0 \in W^{1,p}(0,1)$, $u \in W^{1,p}(0,T)$ with $p \in [1,\infty)$, and the compatibility condition $u(0) = \tilde{p}_0(0)$, as well as $\alpha \in C([0,T],\mathbb{R}^+)$, $\gamma \in C^0(\Omega_T)$, and $\varphi \in C^0([0,T],\mathbb{R})$ hold, then there is a unique solution $\tilde{p}$ satisfying

$$
\tilde{p} \in C^0([0,T];W^{1,p}(0,1)) \cap C^0([0,1];W^{1,p}(0,T)).
$$

Proof. Let us shortly describe the strategy of the proof. First we assume sufficient regularity of a solution (and we assume the existence of a solution) and differentiate the PDE with respect to the spatial variable.

We obtain the linear transport equation (5.7) in $\tilde{p}_x$. Computing the corresponding initial and boundary values, we end up with the initial boundary value problem (5.8)–(5.10). The solution of this problem is determined by the methods of characteristics. Although this method seems to be a straightforward approach, problems arise due to the boundary value in (5.10), where the solution $\tilde{p}_x$, denoted by $q$, reemerges as boundary data.

Reformulating the solution given by the characteristics, we obtain a Volterra integral equation of second kind in $f$ as stated in (5.17), which admits a unique solution. This solution can be used to make the solution $\tilde{p}_x$ more explicit as done in (5.18). So we have constructed a solution of (5.8)–(5.10) and by integration also of the original initial boundary value problem (5.4)–(5.6). The following calculations prove the postulated regularity of the solution.
Let us elaborate this in more detail: Considering the structure of the adjoint equation (5.4), we identify two source terms. The first term is nonlocal depending also on the entire solution for a certain time \( t \) and results from the convective term in the state equation, whereas the second term enters from the cost functional as well as from the boundary due to the coupling conditions.

Obviously, the entire source term does not depend on \( x \), so assuming sufficient regularity we differentiate the PDE with respect to \( x \) and obtain

\[
\dot{\tilde{p}}_x(t) + \alpha(t)\tilde{p}_{xx}(t, x) = 0.
\]

This is a linear transport equation and for given boundary and initial data we can—using the characteristics—give an explicit solution. As it turns out, in the boundary of this PDE the term \( \int_0^1 \gamma(t, s)\tilde{p}_x(t, s) \, ds \) will again emerge, such that we have to apply a fixed point theorem to overcome this difficulty. For the initial value we have \( \tilde{p}_x(0, x) = \tilde{p}'_0(x) \), and for the boundary value we use the PDE to get

\[
\tilde{p}_x(t, 0) = -\frac{1}{\alpha(t)} \left( \frac{1}{\alpha(t)} u'(t) + \int_0^1 \gamma(t, s)\tilde{p}_x(t, s) \, ds + \varphi(t) \right) = -\frac{1}{\alpha(t)} \left( u'(t) + \int_0^1 \gamma(t, s)\tilde{p}_x(t, s) \, ds + \varphi(t) \right).
\]

In this way the nonlocal component of the adjoint equation is shifted into the boundary data. For a clear presentation of the system under consideration, we substitute \( q(t, x) := \tilde{p}_x(t, x), \ (t, x) \in \Omega_T \) to obtain

\[
\dot{q}(t, x) + \alpha(t)q_x(t, x) = 0, \\
q(0, x) = \tilde{p}'_0(x), \quad (t, x) \in \Omega_T, \\
q(t, 0) = -\frac{1}{\alpha(t)} \left( u'(t) + \int_0^1 \gamma(t, s)q(t, s) \, ds + \varphi(t) \right).
\]

Taking the characteristics into account, for \( t \leq \xi^{-1}(1) \), the solution to (5.8)–(5.10) can be expressed via initial and boundary values as (see also Lemma 4.6)

\[
q(t, x) = \tilde{p}_x(t, x) = \begin{cases} q(\tau(t, x), 0) & \text{for } x < \xi(t), \\ \tilde{p}'_0(q(t, x)) & \text{for } x \geq \xi(t). \end{cases}
\]

For \( t > \xi^{-1}(1) \), the solution \( q \) depends only on the boundary conditions, so then we have \( q(t, x) = q(\tau(t, x), 0) \) for all \( x \in [0, 1] \). Here, we only consider the case \( t \leq \xi^{-1}(1) \).

It is important to emphasize that this expression includes an implicit definition of the solution, since the boundary data depends simultaneously on the entire solution at each point in time. In order to resolve this relation, we aim at a different representation which ought to be more convenient to analyze.

Therefore, we consider the source term of the adjoint equation (5.4) for a certain time \( t \in [0, \xi^{-1}(1)] \):}

\[
f(t) := \int_0^1 \gamma(t, s)\tilde{p}_x(t, s) \, ds + \varphi(t).
\]

Moreover, we can split the integral into two individual parts referring to the corresponding division of the explicit solution in (5.11), where \( \xi \) is the characteristic
emanating from the origin as defined in (4.1):

\[ f(t) = \int_0^{\xi(t)} \gamma(s, t) \tilde{p}_x(t, s) \, ds + \int_{\xi(t)}^1 \gamma(t, s) \tilde{p}_x(t, s) \, ds + \varphi(t). \]

Since \( \tilde{p}_x(t, x) = q(t, x) \), we have, for \( 0 \leq s \leq \xi(t) \), \( \tilde{p}_x(t, s) = \tilde{p}_x(\tau(t, s), 0) \), while for \( \xi(t) \leq s \leq 1 \) the characteristic is traced back to the part accounting for the initial data \( \tilde{p}_x(t, s) = \tilde{p}_0^\alpha(\eta(t, s)) \). Inserting these into the integrals gives

(5.13) \[ f(t) = \int_0^{\xi(t)} \gamma(t, s) \tilde{p}_x(\tau(t, s), 0) \, ds + \int_{\xi(t)}^1 \gamma(t, s) \tilde{p}_0^\alpha(\eta(t, s)) \, ds + \varphi(t). \]

In the first integral we substitute \( s(t) = \int_t^\tau \alpha(r) \, dr \) to obtain

(5.14) \[ \int_0^{\xi(t)} \gamma(t, s) \tilde{p}_x(\tau(t, s), 0) \, ds = -\int_0^t \alpha(t) \gamma\left(t, \int_t^\tau \alpha(r) \, dr\right) \tilde{p}_x\left(\tau\left(t, \int_t^\tau \alpha(r) \, dr\right), 0\right) \, dt \]

using \( s(t) = 0 \), \( s(0) = \xi(t) \) and \( \tau(t, \int_t^\tau \alpha(r) \, dr) = \tau(t, \xi(t) - \xi(t)) = t \).

On the other hand, evaluating the adjoint equation (5.4) at \( x = 0 \) and employing the boundary data (5.6) gives

\[ u'(t) + \alpha(t) \tilde{p}_x(t, 0) + \int_0^1 \gamma(t, s) \tilde{p}_x(t, s) \, ds + \varphi(t) = 0 \]

for all \( t \in [0, T] \). Therefore, we can substitute in (5.14) for \( t \in [0, \xi(t)] \) \( \alpha(t) \tilde{p}_x(t, 0) = -u'(t) - \int_0^t \gamma(t, s) \tilde{p}_x(t, s) \, ds - \varphi(t) \) on the right-hand side, and hence

\[ \int_0^{\xi(t)} \gamma(t, s) \tilde{p}_x(\tau(t, s), 0) \, ds = -\int_0^t \gamma\left(t, \int_t^\tau \alpha(r) \, dr\right) \left[u'(t) + \int_0^1 \gamma(t, s) \tilde{p}_x(t, s) \, ds + \varphi(t)\right] \, dt \]

\[ = -\int_0^t \gamma\left(t, \int_t^\tau \alpha(r) \, dr\right) [u'(t) + f(t)] \, dt. \]

Turning toward the second integral in (5.13), we use the transformation \( s(s) = \varsigma(t) \) to shift the integral along the characteristics

\[ \int_{\xi(t)}^1 \gamma(t, s) \tilde{p}_0^\alpha(\eta(t, s)) \, ds = \int_0^{1-\xi(t)} \gamma(t, \varsigma(t) + \xi(t)) \tilde{p}_0^\alpha(\eta(t, \varsigma(t) + \xi(t))) \, ds = \int_0^{1-\xi(t)} \gamma(t, \varsigma(t)) \tilde{p}_0^\alpha(s) \, ds \]

using \( s(0) = \xi(t) \), \( s(1 - \xi(t)) = 1 \), and \( \eta(t, \varsigma(t) + \xi(t)) = \varsigma(t) + \xi(t) - \xi(t) = \varsigma(t) \) (see Definition 4.2 and Remark 4.3). If we put this together, we obtain

\[ f(t) = -\int_0^t \gamma\left(t, \int_t^\tau \alpha(r) \, dr\right) u'(t) \, dt + \int_0^{1-\xi(t)} \gamma(t, \varsigma(t)) \tilde{p}_0^\alpha(s) \, ds \]

\[ -\int_0^t \gamma\left(t, \int_t^\tau \alpha(r) \, dr\right) f(t) \, dt + \varphi(t) = g(t) + \int_0^t k(t, f(t) \, dt \]

with

(5.15) \[ f(t) := \int_0^1 \gamma(t, s) \tilde{p}_x(t, s) \, ds + \varphi(t), \quad k(t, t) := -\gamma\left(t, \int_t^\tau \alpha(r) \, dr\right), \]
(5.16) \( g(t) := -\int_{0}^{t} \gamma \left( t, \int_{t}^{\xi(t)} \alpha(r) \, dr \right) u'(t) \, dt + \varphi(t) + \int_{0}^{1-\xi(t)} \gamma(t, s + \xi(t)) \hat{p}_{0}(s) \, ds \).

Note that \( k \in C^{0}([0, T]) \) since \( t \to \int_{t}^{t} \alpha(r) \, dr \in C^{0}([0, T]) \) and \( \gamma \in C^{0}(\overline{\Omega_{T}}) \) as assumed in the theorem (see Remark 4.3). Furthermore, reconsidering the properties of \( \hat{p}_{0} \) and \( u \), we have \( g \in C^{0}([0, T]) \). So the integral equation

(5.17) \[ f(t) = g(t) + \int_{0}^{t} k(t, t) f(t) \, dt \]

is a Volterra integral equation of second kind, which admits a unique solution \( f \in C^{0}([0, T]) \) (for instance, see [Hac95, Theorem 2.2.1]). Conclusively, we can write the solution \( \hat{p} \), defined in Theorem 5.3, in the following way:

(5.18) \[ \hat{p}(t, x) = \begin{cases} u(\tau(t, x)) - \int_{\tau(t, x)}^{t} f(s) \, ds & \text{for } x \leq \xi(t), \\ \hat{p}_{0}(\eta(t, x)) - \int_{0}^{t} f(s) \, ds & \text{for } x \geq \xi(t). \end{cases} \]

Since we have \( \hat{p}_{0} \in W^{1,p}(0, 1), u \in W^{1,p}(0, T), \) and \( f \in C^{0}([0, T]) \), this can be verified directly. Considering

\[ \hat{p}(t, 0) = u(\tau(t, 0)) + \int_{\tau(t, 0)}^{t} f(s) \, ds = u(t) + \int_{t}^{t} f(s) \, ds = u(t) \]

with \( \tau(t, 0) = t \) shows that the boundary value is attained. Furthermore, we compute the initial value \( \hat{p}(0, x) = \hat{p}_{0}(\eta(0, x)) + \int_{0}^{0} f(s) \, ds = \hat{p}_{0}(x) \), using \( \eta(0, x) = x \). For \( x \leq \xi(t) \) we have

\[
\begin{align*}
\hat{p}(t, x) + \alpha(t) \hat{p}_{x}(t, x) &= u'(\tau(t, x)) \frac{\partial \tau(t, x)}{\partial t} - f(t) + f(\tau(t, x)) \frac{\partial \tau(t, x)}{\partial t} \\
&\quad + \alpha(t) \left( u'(\tau(t, x)) \frac{\partial \tau(t, x)}{\partial x} + f(\tau(t, x)) \frac{\partial \tau(t, x)}{\partial x} \right) \\
&= -f(t) = -\int_{0}^{1} \gamma(t, s) \hat{p}_{x}(t, s) \, ds - \varphi(t),
\end{align*}
\]

where we used (4.6) and (4.7), respectively. For \( x \geq \xi(t) \) we compute

\[
\begin{align*}
\hat{p}(t, x) + \alpha(t) \hat{p}_{x}(t, x) &= \hat{p}_{0}(\eta(t, x)) \frac{\partial \eta(t, x)}{\partial t} - f(t) + \alpha(t) \cdot \hat{p}_{0}'(\eta(t, x)) \frac{\partial \eta(t, x)}{\partial x} \\
&= -f(t) = -\int_{0}^{1} \gamma(t, s) \hat{p}_{x}(t, s) \, ds - \varphi(t),
\end{align*}
\]

where we used (4.5). So the PDE with initial and boundary values holds.

Finally, we show the regularity \( \hat{p} \in C^{0}([0, T]; W^{1,p}(0, 1)) \cap C^{0}([0, 1]; W^{1,p}(0, T)) \), using the explicit solution of \( \hat{p} \) stated in (5.18). For \( t, \xi \in [0, \xi^{-1}(1)] \) and without loss of generality \( t > \xi \) and therefore \( \xi(t) \leq \xi(t) \) (\( \xi \) is a monotone function), we perform the following technical estimation, using Minkowski’s inequality:

\[
\begin{align*}
\|\hat{p}(t, \cdot) - \hat{p}(\xi(t), \cdot)\|_{W^{1,p}(0, 1)} &= \left( \int_{\tau(t, x)}^{\xi(t)} \left| u(\tau(t, x)) - u(\tau(t, x)) \right| f(s) \, ds \\
&\quad + \int_{\xi(t)}^{\xi(t)} f(s) \, ds \right)^{p} \, dx + \int_{\xi(t)}^{\xi(t)} \left| \hat{p}_{0}(\eta(t, x)) - \int_{0}^{t} f(s) \, ds - u(\tau(t, x)) \right| \, dx.
\end{align*}
\]
bounded. Furthermore, for respect to initial and boundary data since

Since $\tilde{p}$ is also an element in $C^0([0,1];W^{1,p}(0,1))$ can be proved by using the PDE or doing similar estimations as for the $C^0([0,T];W^{1,p}(0,1))$-norm.

So we have established that the solution $\tilde{p}$ has the regularity claimed in the theorem.

Remark 5.4 (well-posedness of the optimality system). Let us remark at this point that we now have established the well-posedness of the optimality system in $H^1$. Starting with a given inflow $\bar{\omega}^0$ and $\bar{u}^0$ of regularity $H^1$ we can solve the forward PDE and obtain as right-hand side boundary and end data again $H^1$-regularity. This boundary term will be used as initial and boundary term for the adjoint PDE, which will again produce updates for the forward PDE of sufficient regularity by using, for instance, a gradient scheme as proposed in [LAHR10].

For the problem and the assumptions stated in Theorem 5.3, we establish stability results with respect to the initial and boundary data in the $C([0,T];W^{1,1}(0,1))$-norm. A similar result can be stated for the $C([0,T];W^{1,1}(0,1))$-norm analogously.

Theorem 5.5 (estimates and stability of the solution to the adjoint problem with respect to initial and boundary data). Let $\tilde{p}$ be the solution of problem (5.4)-(5.6).
Then we obtain the estimate
\[
\|\tilde{p}\|_{C([0,T]; W^{1,1}(0,1))} \leq \left( \max \left\{ \alpha_{\text{max}}, \frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \right\} + (2 + \alpha_{\text{max}}) c_1(T, \gamma) \right) \|u\|_{W^{1,1}(0,T)}
\]
\[+ \left( 1 + (2 + \alpha_{\text{max}}) c_1(T, \gamma) \right) \|p_0\|_{W^{1,1}(0,1)} + (2 + \alpha_{\text{max}}) c_2(T, \gamma) \|\varphi\|_{L^\infty(0,T)},
\]
where \(\alpha_{\text{min}} := \min_{t \in [0,T]} \alpha(t) > 0, \alpha_{\text{max}} := \max_{t \in [0,T]} \alpha(t) < \infty\) and
\[
c_1(T, \gamma) := T \max \left\{ \|\gamma\|_{L^\infty(\Omega_T)}, \|\gamma\|_{L^2(\Omega_T)}^2 T \exp(\|\gamma\|_{L^\infty(\Omega_T)} T) \right\},
\]
\[
c_2(T, \gamma) := T^2 \|\gamma\|_{L^\infty(\Omega_T)} \exp(T \|\gamma\|_{L^\infty(\Omega_T)}) + T.
\]

**Proof.** First, we estimate \(f\) using the Volterra integral equation (5.17) and \(f, g, k, \alpha, \varphi\) defined in the proof of Theorem 5.3:
\[
|f(t)| \leq |g(t)| + \int_0^t |k(t, s)| |f(t)| \, ds \leq |g(t)| + \int_0^t \max_{t \in [0,T]} |k(t, s)| |f(t)| \, ds \quad \forall t \in [0,T].
\]

Then we obtain with the help of Gronwall’s inequality a qualitative estimate on \(|f(t)|\):
\[
|f(t)| \leq |g(t)| + \int_0^t |g(t)| \max_{t \in [0,T]} |k(t, s)| \exp \left( \int_0^t \max_{t \in [0,T]} |k(t, s)| \, ds \right) \, dt
\]
\[
\leq |g(t)| + \|k\|_{L^\infty([0,T]^2)} \int_0^t |g(t)| \exp \left( \|k\|_{L^\infty([0,T]^2)} (t - s) \right) \, ds
\]
\[
\leq |g(t)| + \|k\|_{L^\infty([0,T]^2)} \exp \left( t \|k\|_{L^\infty([0,T]^2)} \right) \int_0^t |g(t)| \, ds.
\]

Inserting the definition in (5.16) for \(g\) and (5.15) for \(k\), we obtain
\[
|f(t)| \leq \left| \int_0^{1-\xi(t)} \gamma(t, s + \xi(t)) \tilde{p}_0(s) \, ds - \int_0^t \gamma(t, \int_s^t \alpha(r) \, dr) u'(t) \, dt + \varphi(t) \right|
\]
\[
+ \|\gamma\|_{L^\infty(\Omega_T)} \exp \left( t \|\gamma\|_{L^\infty(\Omega_T)} \right) \int_0^t \int_s^{1-\xi(t)} \gamma(t, s + \xi(t)) \tilde{p}_0'(s) \, ds
\]
\[
- \int_s^{1-\xi(t)} \gamma(t, \int_s^t \alpha(r) \, dr) u'(s) \, ds + \varphi(t) \right| \, dt
\]
\[
\leq \|\gamma\|_{L^\infty(\Omega_T)} \int_0^{1-\xi(t)} |\tilde{p}_0'(s)| \, ds + \|\gamma\|_{L^\infty(\Omega_T)} \int_0^t \left| u'(t) \right| \, dt + \|\varphi\|_{L^\infty(0,T)}
\]
\[
+ \|\gamma\|_{L^\infty(\Omega_T)}^2 \exp \left( t \|\gamma\|_{L^\infty(\Omega_T)} \right) \int_0^t \left( \int_s^{1-\xi(t)} |\tilde{p}_0'(s)| \, ds + \int_0^t |u'(s)| \, ds \right) \, dt
\]
\[
+ \|\gamma\|_{L^\infty(\Omega_T)} \exp \left( t \|\gamma\|_{L^\infty(\Omega_T)} \right) \|\varphi\|_{L^\infty(0,T)}
\]
\[
\leq \|\gamma\|_{L^\infty(\Omega_T)} \int_0^t |\tilde{p}_0'(s)| \, ds + \|\gamma\|_{L^\infty(\Omega_T)} \int_0^T \left| u'(t) \right| \, dt
\]
\[
+ \|\gamma\|_{L^\infty(\Omega_T)}^2 \exp \left( t \|\gamma\|_{L^\infty(\Omega_T)} \right) \int_0^t \left( \int_s^1 |\tilde{p}_0'(s)| \, ds + \int_0^t |u'(s)| \, ds \right) \, dt
\]
\[
+ \|\gamma\|_{L^\infty(\Omega_T)} \exp \left( t \|\gamma\|_{L^\infty(\Omega_T)} \right) \|\varphi\|_{L^\infty(0,T)} + \|\varphi\|_{L^\infty(0,T)}
\]
\[ \leq \| \gamma \| L^\infty(\Omega_T) \| p_0 \| L^1(0,1) + \| \gamma \| L^\infty(\Omega_T) \| u' \| L^1(0,T) \\
+ \| \gamma \| L^2(\Omega_T) \exp\left( T \| \gamma \| L^\infty(\Omega_T) \right) T \left( \| p_0 \| L^1(0,1) + \| u' \| L^1(0,T) \right) \\
+ \left( \| \gamma \| L^\infty(\Omega_T) \exp\left( T \| \gamma \| L^\infty(\Omega_T) \right) T + 1 \right) \| \varphi \| L^\infty(0,T). \]

So we have

\[
\| f \| L^1(0,T) \leq T \max \left\{ \| \gamma \| L^\infty(\Omega_T), \| \gamma \| L^2(\Omega_T) \exp\left( T \| \gamma \| L^\infty(\Omega_T) \right) T \right\},
\]

(5.19)

\[
\left( \| p_0 \| W^{1,1}(0,1) + \| u \| W^{1,1}(0,T) \right) + \left( \| \gamma \| L^\infty(\Omega_T) \exp\left( T \| \gamma \| L^\infty(\Omega_T) \right) T + 1 \right) \| \varphi \| L^\infty(0,T) \\
= c_1(T, \gamma) \left( \| p_0 \| W^{1,1}(0,1) + \| u \| W^{1,1}(0,T) \right) + c_2(T, \gamma) \| \varphi \| L^\infty(0,T)
\]

with \( c_1(T, \gamma) := T \max \{ \| \gamma \| L^\infty(\Omega_T), \| \gamma \| L^2(\Omega_T) \exp\left( T \| \gamma \| L^\infty(\Omega_T) \right) T \} \) and \( c_2(T, \gamma) := T^2 \| \gamma \| L^\infty(\Omega_T) \exp\left( T \| \gamma \| L^\infty(\Omega_T) \right) T + T \). Second, one can estimate

\[
\| \tilde{p} \| C([0,T], W^{1,1}(0,1)) = \max_{t \in [0,T]} \left( \left\| u(t, \cdot) - \int_{\tau(t, \cdot)}^{t} f(s) \, ds \right\|_{L^1(0,\xi(t))} \\
+ \left\| \tilde{p}_0(\eta(t, \cdot)) - \int_{0}^{t} f(s) \, ds \right\|_{L^1(\xi(t),1)} + \left\| u'(\tau(t, \cdot)) \right\|_{L^1(0,\xi(t))} \right) \\
+ \left( \left\| \tilde{p}_0(\eta(t, \cdot)) \right\|_{L^1(\xi(t),1)} \right)
\leq \max_{t \in [0,T]} \left( \left\| u(t, \cdot) \right\|_{L^1(0,\xi(t))} + \left\| \int_{\tau(t, \cdot)}^{t} f(s) \, ds \right\|_{L^1(0,\xi(t))} \\
+ \left\| \tilde{p}_0(\eta(t, \cdot)) \right\|_{L^1(\xi(t),1)} \right) + \left\| f(\tau(t, \cdot)) \right\|_{L^1(0,\xi(t))} \\
+ \frac{1}{\alpha_{\min}} \left\| u'(\tau(t, \cdot)) \right\|_{L^1(0,\xi(t))} + \left\| f(\tau(t, \cdot)) \right\|_{L^1(0,\xi(t))} \\
\leq \alpha_{\max} \| u \| L^1(0,T) + \| f \| L^1(0,T) + \frac{\alpha_{\max}}{\alpha_{\min}} \| u' \| L^1(0,T) \\
+ \alpha_{\max} \| f \| L^1(0,T) + \| \tilde{p}_0 \| L^1(0,1) + \| f \| L^1(0,T) + \| \tilde{p}_0 \| L^1(0,1) \\
\leq \max \left\{ \alpha_{\max}, \frac{\alpha_{\max}}{\alpha_{\min}} \right\} \| u \| W^{1,1}(0,T) + (2 + \alpha_{\max}) \| f \| L^1(0,T) + \| \tilde{p}_0 \| W^{1,1}(0,1).
\]

Now using the inequality (5.19) for \( \| f \| L^1(0,T) \), we obtain

\[
\| \tilde{p} \| C([0,T], W^{1,1}(0,1)) \leq \max \left\{ \alpha_{\max}, \frac{\alpha_{\max}}{\alpha_{\min}} \right\} \| u \| W^{1,1}(0,T) + \| \tilde{p}_0 \| W^{1,1}(0,1) \\
+ (2 + \alpha_{\max})c_1(T, \gamma) \left( \| \tilde{p}_0 \| W^{1,1}(0,1) + \| u \| W^{1,1}(0,T) \right) + (2 + \alpha_{\max})c_2(T, \gamma) \| \varphi \| L^\infty(0,T) \\
= \left( \max \left\{ \alpha_{\max}, \frac{\alpha_{\max}}{\alpha_{\min}} \right\} \right) + (2 + \alpha_{\max})c_1(T, \gamma) \| u \| W^{1,1}(0,T) \\
+ (1 + (2 + \alpha_{\max})c_1(T, \gamma)) \| \tilde{p}_0 \| W^{1,1}(0,1) + (2 + \alpha_{\max})c_2(T, \gamma) \| \varphi \| L^\infty(0,T).
\]

A similar result for \( p > 1 \) can be proved. \( \qed \)
6. Conclusion. In this article we consider an optimal boundary control problem subject to a first order nonlinear, nonlocal hyperbolic PDE. We establish existence, uniqueness, and regularity properties of the solutions in a general $W^{1,p}$-setting. Based on a general cost functional, the special structure of the corresponding optimality system is discussed. Due to the nonlocal dependence of the flux function on the current solution, we identify necessary compatibility conditions and demonstrate the need of an appropriate choice of the cost functional. In a subsequent step, the existence, uniqueness, and regularity of the adjoint system is derived for $W^{1,p}$-data justifying the introduced Lagrange formalism. The obtained results are finally completed by a proof of the well-posedness with respect to the initial data.

The established theory covers many applications ranging from supply chain management to particle synthesis processes. The results obtained give rise to many interesting question concerned with a relaxation of the presumed regularity in a rigorous treatment of this nonlocal setting.

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